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A MATHEMATICAL TREATMENT OF ONE-DIMENSIONAL SOIL CONSOLIDATION*

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Summary. Terzaghi's conception of the nature of one-dimensional soil consolidation [1] is shown to lead to a non-linear differential equation. A dimensional analysis of this equation and the boundary conditions of the standard consolidation test [2] gives a more general explanation of a well known linear relationship between the total consolidation $U(t)$ after a time t and $t^{1/2}$. By linearizing the equation in a general manner, an expression is obtained for $U(t)$ which includes secondary consolidation terms. Two solutions of the linearized equation are obtained; the first for the standard consolidation test and the second for consolidation under a boundary load increasing uniformly with time.

Introduction. A medium of clay saturated with water and subjected to a constant pressure, continues to contract in volume over a long period of time. Terzaghi, in his *Erdbaumechanik* [3], put forward a diffusion theory to explain the time dependent nature of this volume contraction. The following three ideas are the basis of his treatment and will be taken as the starting point of this paper.

(1) The medium is imagined as a porous water saturated skeleton of particles of negligible compressibility, so that a change in the volume of any region of the skeleton equals the volume of water displaced through its boundary.

(2) Stresses applied to the boundary of the clay produce a hydrostatic head h in the pore water and a stress at each point of the skeleton of such a direction and magnitude as to maintain static equilibrium at all instants of time.

(3) It is finally assumed that spatial variations in h cause the pore water to diffuse through the skeleton in accordance with Darcy's law, which equates the quantity of water flowing through any small plane surface σ in unit time to the product of the area of σ , the rate of change of h with distance along the normal to σ and a constant K called the permeability.

By considering the problem of the consolidation of clay in one direction only, the need for a tensor description of the stress distribution is avoided. The one-dimension problem is defined by the assumption that the fluid flow and the motion of the skeleton particles are in one direction only and that all relevant physical properties are constant at any instant of time on any plane with a normal along this direction. The stress function is then the pressure $P(z, t)$ which the skeleton in the neighborhood of the plane z can support at time t and, like K , will depend on the physical state of the solid matter

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of the clay. If the skeleton is a homogeneous aggregate of small non-spherical particles, the hydrological state of clay is described by two functions. The first, called the void ratio ϵ , and defined as the volume of water saturating unit volume of the porous structure in the neighborhood of the point, describes the degree of compaction of the aggregate, while the second, a distribution function ϕ , describes the orientation of the particles with respect to the direction of consolidation. It is evident that ϕ will depend on the previous history of the medium but it will be assumed to begin with that this is such that K can be treated as a function of ϵ only.

By assuming a linear relationship between ϵ and P and a constant value for K , Terzaghi derived an equation of the form

$$K \frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial t}$$

for P .

Let us, however, derive the general mathematical formulation of the ideas outlined above and make approximations only when compelled to by the difficulty of solving specific problems.

The equation of one-dimensional consolidation theory. Let us define z to be the volume of the skeleton of the clay bounded by a cylinder S of unit cross-sectional area with its axis in the direction of consolidation, and the two planes $0, z$ fixed relative to the skeleton. If $L(z, t)$ is the distance between these two planes at time t ,

$$\frac{\partial L}{\partial z} = 1 + \epsilon \quad \text{since} \quad L = \int_0^z (1 + \epsilon) dz.$$

The quantity of water $Q(t)$ in this region is given by the expression

$$Q = \int_0^z \epsilon dz \quad (1)$$

and hence the rate at which water flows through the planes $0, z$ into the region is $\partial Q / \partial t$.

Now by Darcy's law,

$$\frac{\partial Q}{\partial t} = \left[K \frac{\partial h}{\partial z} / \frac{\partial L}{\partial z} \right]_0^z \quad (2)$$

and so

$$\int_0^z \frac{\partial \epsilon}{\partial t} dz = \left[\frac{K}{1 + \epsilon} \frac{\partial h}{\partial z} \right]_0^z.$$

The derivative of this equation with respect to z is

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z} \left(\frac{K}{1 + \epsilon} \frac{\partial h}{\partial z} \right).$$

Suppose $P_0(t)$ is the pressure on two planes z_1, z_2 , say, bounding the clay. Since consolidation is slow, the total internal pressure is also $P_0(t)$. Hence

$$P(z, t) + h(z, t) = P_0(t),$$

and the derivative of this equation with respect to z ,

$$\frac{\partial P}{\partial z} = -\frac{\partial h}{\partial z}$$

inserted in (2) gives the one-dimensional consolidation equation in its final form;

$$\frac{\partial \epsilon}{\partial t} = -\frac{\partial}{\partial z} \left(\frac{K}{1 + \epsilon} \frac{\partial P}{\partial z} \right). \quad (3)$$

Two more equations connecting K , ϵ and P and a complete set of boundary conditions are needed to specify a soluble mathematical problem. The complexity of the geometry makes the problem of predicting the nature of these relationships a difficult one, and little can be said beyond the intuitively obvious statement that K tends to zero and P increases when ϵ decreases.

It is shown in the next section that, provided these two relationships do not explicitly involve z or t , the system can be solved for the case of the consolidation of a semi-infinite medium under a constant boundary pressure. In this case a dimensional analysis shows that the volume of water displaced from the skeleton is proportional to the square root of the duration of the consolidation.

The square root law. If the functions K and P are dependent on the void ratio ϵ alone, Eq. (3) is invariant under the family of similarity transformations

$$z \rightarrow \alpha z, \quad t \rightarrow \alpha^2 t, \quad \alpha > 0.$$

The boundary conditions associated with the problem of the consolidation of a semi-infinite medium under a constant boundary pressure P_0 are:

$$\epsilon = \epsilon_0 \text{ at } t = 0 \text{ for all } z > 0$$

and

$$P = P_0 \text{ at } z = 0 \text{ for all } t > 0.$$

These conditions are also invariant under each member of the set of transformations above. If the functions $K(\epsilon)$ and $P(\epsilon)$ are such that the boundary conditions uniquely determine the solution of (3), the functions $\epsilon(z, t)$ and $\epsilon(\alpha z, \alpha^2 t)$ must be identically equal for all $\alpha > 0$ in the region of positive values of z and t . This implied that ϵ is a function of $z/t^{1/2}$ only, for by writing z^* for αz and t^* for $\alpha^2 t$ and observing that $\epsilon(z^*, t^*)$ is independent of α for positive z and t , since in this region it is identically equal to $\epsilon(z, t)$, we obtain by differentiation, the equation

$$\partial \epsilon / \partial \alpha = z \partial \epsilon / \partial z^* + 2\alpha t \partial \epsilon / \partial t^* = 0.$$

This equation, when multiplied by α , gives the linear partial differential equation

$$z^* \partial \epsilon / \partial z^* + 2t^* \partial \epsilon / \partial t^* = 0$$

with the general solution

$$\epsilon = f(z^*/t^{1/2}) = f(z/t^{1/2})$$

involving the arbitrary function f .

Let us write $y = z/t^{1/2}$, $\epsilon = f(y)$. Since

$$\partial \epsilon / \partial t = (\partial f / \partial y)(\partial y / \partial t) = -y/2t^{3/2} \cdot df/dy$$

and

$$\partial P / \partial z = 1/t^{1/2} \cdot dP/dy$$

Eq. (3) reduces to the ordinary differential equation

$$\frac{y}{2} \frac{df}{dy} = \frac{d}{dy} \left(\frac{K}{1 + \epsilon} \frac{dP}{dy} \right) \quad (4)$$

while the boundary conditions become:

$$P = P_0 \text{ at } y = 0$$

and ϵ tends to ϵ_0 as y tends to infinity.

The progress of the consolidation is measured by the volume of water which has seeped through each unit of area of the surface $z = 0$ after the application of the boundary pressure P_0 . The total consolidation $U(t)$ in this problem is the limiting value of $L(z, 0) - L(z, t)$ as z tends to infinity and hence

$$\begin{aligned} dU/dt &= \lim_{z \rightarrow \infty} -\partial L / \partial t \\ &= \lim_{z \rightarrow \infty} -\partial Q / \partial t \\ &= \left[\frac{K}{1 + \epsilon} \frac{\partial P}{\partial z} \right]_{z=0} = \frac{K_0}{1 + \epsilon_0} \left(\frac{dP}{dy} \right)_{y=0} \cdot \frac{1}{t^{1/2}} \end{aligned}$$

by a direct substitution from Eq. (2).

$K_0/(1 + \epsilon_0)$ depends only on the value ϵ_0 of ϵ which satisfies the boundary condition $P(\epsilon) = P_0$, while $(dP/dy)_{y=0}$ is a constant found by solving Eq. (4) and evaluating $dP/d\epsilon \cdot df/dy$ at $y = 0$. The differential equation for $U(t)$ can be easily integrated with the boundary condition $U = 0$ at $t = 0$ to give the result

$$U = 2At^{1/2}, \text{ where } A \text{ is the constant } \left[\frac{K}{1 + \epsilon} \frac{dP}{dy} \right]_{y=0}. \quad (5)$$

This result is of interest since the semi-infinite problem is closely related to the well known consolidation problem with the modified boundary conditions

$$\epsilon = \epsilon_0 \text{ or } P = P'_0 \equiv P(\epsilon_0) \text{ at } t = 0 \text{ for } 0 < z < z_1$$

and

$$P = P_0 \text{ for all } t > 0 \text{ on the planes } z = 0 \text{ and } z_1.$$

During the initial stages of consolidation, in fact till $P(z_1/2t^{1/2})$ differs appreciably from P'_0 , this is essentially the superposition of two semi-infinite cases; and hence $U(t)$ is given initially by $4At^{1/2}$.

Terzaghi's linear theory of course, also predicts this result as it corresponds to a very special case of Eq. (3). Agreement between his formula for $U(t)$ and the experimental consolidation curves over the initial stages of compression has no bearing on the validity of the assumptions he has used in linearizing the theory. In many cases, agreement between his theory and experiment is in this region of linearity only, and the conclusion to be drawn is that in this region K and P are effectively functions of ϵ only.

The experimental determination of $K(\epsilon)$ and $P(\epsilon)$. If P is a function of ϵ only, there is no great difficulty in determining experimentally the functional relationship between them. This is usually done by carrying out a sequence of standard consolidation tests [2] and calculating the value of the void ratio at the conclusion of each test. The problem of determining the relationship between K and ϵ is a more difficult one.

The constant A in Eq. (5) seems at first to be proportional to $K(\epsilon_0)$. However, this is not so, because $(dP/dy)_{y=0}$ is indirectly related to the function $K(\epsilon)$ by the differential equation (4). This equation is insoluble except for simple forms of the functions f and k and these forms do not involve enough parameters to describe reasonably the functions. An empirical approach which partly avoids solving Eq. (4) is outlined below.

Suppose a sequence of standard consolidation tests is carried out on the same sample $0 < z < z_1$ and the load applied to the boundary is increased by a factor β at each step. If each experiment is carried through to the completion of the primary consolidation, and secondary consolidation is negligible, the boundary conditions for the r th test are;

$$P = \beta^{r-1} P_0 \text{ at } t = 0 \text{ for } 0 < z < z_1$$

$$P = \beta^r P_0 \text{ at } z = 0 \text{ and } z = z_1 \text{ for } t > 0.$$

The boundary conditions for the corresponding semi-infinite problem giving the gradient of the $(U, t^{1/2})$ line are

$$P \rightarrow \beta^{r-1} P_0 \text{ as } y \text{ tends to infinity and } P = \beta^r P_0 \text{ at } y = 0.$$

A measurement of ϵ at the conclusion of each test gives a set of points $(\epsilon_r, \beta^r P_0)$ which map the relationship between P and ϵ . Let us choose the values of the three constants C' , C'' , m in the empirical relation,

$$\epsilon = C'P^m + C'' \quad (6)$$

which gives the closest fit to the experimental curve.

If it is assumed that $K/(1 + \epsilon)$ can be approximated by an equation with two arbitrary constants C , n of the form

$$K/(1 + \epsilon) = CP^n, \quad (7)$$

the auxiliary equation for determining A_r , and so the gradient of the $(U, t^{1/2})$ curve for the r th test is

$$\frac{y}{2} \frac{d}{dy} (C'P^m) = \frac{d}{dy} \left(CP^n \frac{dP}{dy} \right). \quad (8)$$

Let $P = \beta^{r-1} p^*$ and $y = \beta^{(r-1)(n-m+1)/2} y'^*$ so that the equation determining A_r becomes

$$\frac{C'}{C} \frac{y^*}{2} \frac{dP^{*m}}{dy^*} = \frac{d}{dy^*} \left(P^{*n} \frac{dP^*}{dy^*} \right) \quad (9)$$

and the boundary conditions in terms of y and z' are

$$P^* \rightarrow P_0 \text{ as } y^* \text{ tends to infinity and } P^* = \beta P_0 \text{ at } y^* = 0.$$

This system is independent of r and so dP^*/dy^* has the same value at $y = 0$ for each value of r . We see from (5) the equation for A_r is

$$\begin{aligned} A_r &= \left[\frac{K}{1 + \epsilon} \frac{dP}{dP^*} \frac{dP^*}{dy^*} \frac{dy^*}{dy} \right]_{y=0} \\ &= CP_0^n \beta^{r(n+m+1)/2 + (n-m-1)/2} \left(\frac{dP^*}{dy^*} \right)_{y^*=0} \end{aligned} \quad (10)$$

or

$$\log A_r = \frac{r(n + m + 1)}{2} \log \beta + \log B,$$

where B is a constant independent of r . A graph of $\log A_r$ against $r \log \beta$ should therefore be a straight line of gradient $(n + m + 1)/2$ for the values of r over which the forms chosen for ϵ and $K/(1 + \epsilon)$ are valid approximations. C must now be found by solving Eq. (9) and it seems that this must be done numerically.

Linear theories of consolidation. The solution of a one-dimensional consolidation problem is determined theoretically by the experimental relationships connecting ϵ , P and K which give an explicit form to (3), and by the appropriate set of boundary conditions. In practice, mathematical difficulties make it necessary to linearize (3) in order to obtain analytical solutions for boundary conditions more complex than those considered above. This had always been done by assuming $K/(1 + \epsilon)$ is constant and choosing the linear relationship between P and ϵ which best fits the experimental curve.

Since there are two functions at our disposal, this linearizing process can be carried out in a less restrictive manner. It can readily be seen that to linearize (3), it is necessary that K , P and ϵ be related by an equation of the form

$$\frac{K}{1 + \epsilon} \frac{\partial P}{\partial z} = -k \frac{\partial \epsilon}{\partial z}, \quad (11)$$

where k is independent of ϵ . Equation (3) and this relationship, together give a linear equation for ϵ of the form

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z} \left(k \frac{\partial \epsilon}{\partial z} \right). \quad (12)$$

If k is constant or a function of t only, (12) can be put in the dimensionless form,

$$\frac{\partial \epsilon}{\partial \tau} = \frac{\partial^2 \epsilon}{\partial x^2}, \quad (13)$$

where $\tau = \int_0^t k/a^2 dt$ and $x = z/a^2$, a being a convenient measure of length. τ will be used as a measure of time in all that follows and " a " will usually be half of the volume of the skeleton between the boundary planes and the cylinder S of unit cross-sectional area normal to these planes.

Equation (11) gives an explicit expression for K

$$K = -k \frac{\partial}{\partial P} \left(\epsilon + \frac{\epsilon^2}{2} \right) \quad (14)$$

which is fortunately not physically unreasonable. ϵ decreases monotonically to a finite limit as P increases and so K , as determined by (14), is positive for positive k and tends to zero as ϵ decreases. There is still freedom to choose ϵ as any function of P and t , as one more relationship between ϵ , K and P is required to specify the consolidation problem mathematically. It is interesting to note how previous consolidation theories comply with (14). Neglecting ϵ^2 and taking ϵ as a linear function of P , and k as a constant, gives a constant value for K as in Terzaghi's theory. The linear theory proposed by D. W. Taylor and W. Marchant in 1940 [4] has a constant K and an $\epsilon - P$ relationship of the form

$$\epsilon = \epsilon_0 - c(P - P_0) + (c - c')(P - P_0) \exp(-\mu t).$$

Equation (14) is satisfied by choosing k a function of t as defined by the equation

$$k(t)[c - (c - c') \exp(-\mu t)] = K$$

and a simple integration gives

$$\tau = \frac{K}{ca^2} \log \left[\frac{c \exp(\mu t) + (c' - c)}{c} \right].$$

Standard consolidation tests on many types of clays show that ϵ is a function of t as well as P , for, when subjected to the constant boundary pressure P_0 , ϵ continues to decrease after the hydrostatic head h should have vanished. Equation (3) is still valid for a time-dependent $\epsilon - P$ relationship and so this behavior, usually described as secondary consolidation, can be incorporated in a linear theory by using a suitable $\epsilon - P - t$ relationship and a corresponding permeability K , defined by (14). The function $\epsilon(P, t)$ for many soils, can be adequately described by a form

$$\epsilon(P, t) = B - B' \log P - B'' \log(1 + \tau) \quad (15)$$

involving three constants B , B' and B'' .

k can be chosen as either that constant which makes the curve

$$K = B'k [1 + B - B' \log P - B'' \log(1 + \tau)]/P$$

fit a known relationship between K and P closely, or that which gives the time scale $\tau = kt/a^2$, which agrees with an experimental curve.

The solution of the linear consolidation equation with general boundary conditions.

Let us consider the one-dimensional consolidation of a homogeneous clay medium bounded by plane surfaces at $z = a$ and $-a$ and the cylinder of unit cross-sectional area, S , normal to these, under a boundary pressure $P_0(\tau)$, which varies during the experiment. We will also suppose there is an initial variation in the void ratio along the axis of the cylinder. In mathematical terms, we wish to find the solution of Eq. (13) subject to the boundary conditions

$$\epsilon = \epsilon_0(x) \text{ at } \tau = 0 \text{ for } x \text{ in the interval } -1 < x < 1$$

and

$$\epsilon = \epsilon[P_0(\tau), \tau] \equiv g(\tau) \text{ say, on } x = \pm 1 \text{ for } \tau > 0.$$

The Laplace transform of the solution is readily obtained by conventional methods and was found to be

$$\epsilon_L = (g_L + I_1) \frac{\cosh qx}{\cosh q} + I_2 \frac{\sinh qx}{\sinh q} - I_3,$$

where

$$I_1 = \int_0^1 \frac{\epsilon_0(x') + \epsilon_0(-x')}{2q} \sinh q(1 - x') dx',$$

$$I_2 = \int_0^1 \frac{\epsilon_0(x') - \epsilon_0(-x')}{2q} \sinh q(1 - x') dx',$$

$$I_3 = \int_0^1 \frac{\epsilon_0(x')}{q} \sinh q(x - x') dx',$$

$$\epsilon_L = \int_0^\infty e^{-s\tau} \epsilon(x, \tau) d\tau, \quad g_L = \int_0^\infty e^{-s\tau} g(\tau) d\tau \quad \text{and} \quad q^2 = s.$$

The total consolidation is by definition $L(a, 0) - L(a, \tau) - L(-a, 0) + L(-a, \tau)$, where

$$L(z, \tau) = \int_0^z (1 + \epsilon) dz$$

and hence

$$U(\tau) = a \int_{-1}^1 (\epsilon_0 - \epsilon) dx$$

while

$$\begin{aligned} U_L &\equiv \int_0^\infty e^{-s\tau} U d\tau = a \int_{-1}^1 \left(\frac{\epsilon_0}{s} - \epsilon_L \right) dx \\ &= 2a \left[\int_0^1 \frac{\epsilon_0(x') + \epsilon_0(-x')}{2s} \frac{\cosh qx'}{\cosh q} dx' - g_L \frac{\tanh q}{q} \right]. \end{aligned} \quad (16)$$

If $\epsilon_0(x)$ is a constant ϵ_0 say, plus an odd function, the first term of (16) simplifies and

$$U_L = 2a \left(\frac{\epsilon_0}{s} - g_L \right) \frac{\tanh q}{q}. \quad (17)$$

In general $\epsilon_0(x)$ affects the behavior of U near $\tau = 0$, especially if the deviation from a constant plus an odd function is greatest near $x = +$ or -1 . In the standard consolidation test, this effect will appear as an initial departure from linearity in the $(U, t^{1/2})$ curve and so as a form of pre-primary consolidation.

The Laplace transform (17) can be readily inverted to give

$$\dot{U} = 2a \int_0^\tau [\epsilon_0 - g(\tau - t)] \Theta_2(0 | i\pi t) dt, \quad (18)$$

where $\Theta_2(\nu | t)$ is the theta function defined p. 388 of [5]. For large values of τ this integral is asymptotic to $2a(\epsilon_0 - g)$ while for small values of τ it behaves like

$$2a\pi^{-1/2} \int_0^\tau [\epsilon_0 - g(\tau - t)] t^{-1/2} dt \quad (19)$$

or like $4a[\epsilon_0 - g(0)](\tau/\pi)^{1/2}$ if $dg/d\tau$ is small. Since $g(\tau) = \epsilon[P_0(\tau), \tau]$, $g(0)$ is the initial void ratio on the boundaries.

The standard consolidation test. The pressure on the boundaries in the standard consolidation test [2] is constant, P_0 , say, and so $g(\tau) = \epsilon(P_0, \tau)$ is a function describing the secondary consolidation. If, for simplicity, we neglect any initial variation in the void ratio, we have the immediate result U behaves like $2a(\epsilon_0 - g)$ for large values of τ , which is for the relationship (15) $2aB'' \log(1 + \tau)$.

Let us consider in more detail, the consolidation of soils adequately described by Eq. (15). We see from Eq. (17) that the Laplace transform of the total consolidation is

given by the expression

$$U_L = 2a \left[\frac{\epsilon_0 - \epsilon_1}{s} - \frac{B''}{s} e^{Ei(-s)} \right] \frac{\tanh q}{q}, \quad (20)$$

where $\epsilon_1 = B - B' \log P_0$.

The first term of (20) is the transform of the well known solution of the standard consolidation problem, while the second gives the effect of secondary consolidation on U_L . These terms can be inverted by well known techniques of the Operational Calculus to give series of asymptotic solutions useful for small values of τ and others for large τ . Let us write U as $U_1 + U_2$, where U_1 is the primary and U_2 the secondary consolidation term.

For small values of τ

$$U_1 = 4a(\epsilon_0 - \epsilon_1) \left\{ \pi^{-1/2} \tau^{1/2} + 2 \sum_{r=1}^{\infty} (-1)^r [\pi^{-1/2} \tau^{1/2} \exp(-r^2/\tau) - r \operatorname{Erfc}(r/\tau^{1/2})] \right\} \quad (21)$$

and

$$U_2 = 8aB''\pi^{-1/2} \{ (1 + \tau)^{1/2} \log [\tau^{1/2} + (1 + \tau)^{1/2}] - \tau^{1/2} \} + 4aB'' \sum_{r=1}^{\infty} (-1)^r J_r, \quad (22)$$

where

$$J_r = \int_0^{\tau} \pi^{-1/2} t^{-1/2} \exp(-r^2/t) \log(1 + \tau - t) dt.$$

As J_r is not expressible in finite terms, the series (22) is not very useful for values of τ for which J_1 cannot be neglected. However, we may use the first term till

$$2 \log(1 + \tau) \{ \pi^{-1/2} \tau^{1/2} \exp(-1/\tau) - \operatorname{erfc}(1/\tau^{1/2}) \}$$

which is an upper bound to J_1 , ceases to be negligible. For large values of the dimensionless parameter τ

$$U_1 = 2a(\epsilon_0 - \epsilon_1) \left\{ 1 - \frac{8}{\pi^2} \sum_{r=0}^{\infty} \exp[-(2r+1)^2 \pi^2 \tau/4] / (2r+1)^2 \right\}, \quad (23)$$

and

$$U_2 \sim 2aB'' \left\{ \log(1 + \tau) + \sum_{r=0}^{\infty} M_r \exp[-\pi^2(2r+1)^2 \tau/4] + N_r / (1 + \tau)^{r+1} \right\},$$

$$M_r = 8\pi^{-2} \exp[-\pi^2(2r+1)^2/4] Ei^*[\pi^2(2r+1)^2/4], \quad (24)$$

$$N_r = (-4)^{r+2} (4^{r+2} - 1) r! B_{2r+4} / (2r+4)!,$$

where B_{2r+4} are Bernoulli numbers and $Ei^*(x)$ is the logarithmic integral defined p. 2 of [6] as $li(e^x)$.

The first few terms of the asymptotic expression (24), which is useful for values of τ outside the range of (22), are

$$U_2 \sim 2aB'' \left\{ \log(1 + \tau) - \frac{1}{3(1 + \tau)} - \frac{2}{15(1 + \tau)^2} - \dots \right. \\ \left. + .4755 \exp(-2.4674\tau) + 18.895 \exp(-22.207\tau) + \dots \right\}.$$

Consolidation under a linearly increasing boundary pressure. It was pointed out earlier, that the standard consolidation test will not effectively justify approximations which are made to linearize Eq. (3). It is evident that in an experiment designed to test an approximation arising from (14), the pressure applied to the boundary planes must vary with time. Let us consider the simplest problem of this type, that with a linearly increasing boundary pressure.

Suppose a pressure P_0 is first applied to the boundaries and maintained constant until the void ratio attains a value ϵ_0 throughout the medium. Now, at $\tau = 0$ let us apply the boundary pressure

$$P_0(\tau) = P_0(1 + b\tau).$$

We see from (15) and (17) that

$$\begin{aligned} U_L &= -2a[B' \exp(s/b)Ei(-s/b) + B'' \exp(s)Ei(-s)] \frac{\tanh q}{sq} \\ &\equiv (U_3)_L + (U_2)_L \text{ say} \end{aligned} \quad (25)$$

U_2 is given directly by (22) and (24) and, by modifications in the derivation of these formulae, it can be shown that, for small values of τ ,

$$U_3 = 8aB'(b\pi)^{-1/2} \{ (1 + b\tau)^{1/2} \log [(b\tau)^{1/2} + (1 + b\tau)^{1/2}] \} + 4aB' \sum_{r=1}^{\infty} (-1)^r J'_r, \quad (26)$$

where

$$J'_r = \pi^{-1/2} \int_0^\tau t^{-1/2} \exp(-r^2/t) \log [1 + b(\tau - t)] dt$$

and J'_1 is bounded above by the expression

$$2 \log(1 + b\tau) [\pi^{-1/2} \tau^{1/2} \exp(-1/\tau) - \operatorname{erfc}(1/\tau^{1/2})],$$

while for large values of τ

$$U_3 \sim \frac{2aB'}{b} \log(1 + b\tau) + \sum_{r=0}^{\infty} M'_r \exp[-(2r+1)^2 \pi^2 \tau / 4] + N_r b' / (1 + b\tau)^{r+1},$$

where N_r is as defined in (24) and

$$M'_r = 8\pi^{-2} \exp[-\pi^2(2r+1)^2/4b] Ei^*[\pi^2(2r+1)^2/4b]. \quad (27)$$

It will be found that the two approximate forms for U_2 and U_3 are unsatisfactory in an interval of values of τ . It would be useful to know J_1 and J'_1 in this interval and they could be calculated by a numerical integration of the integrals defining them, but it would be better in this case to use the appropriate form of (18).

The consolidation of media of compressible particles. It was assumed, in deriving Eq. (3), that the particles composing the skeleton of the clay had a negligible compressibility. This seems to offer one direction in which the theory presented in this paper can be generalized, but it is not difficult to introduce a compressibility factor into the treatment and show that the conclusions and formulae come out practically unchanged. If z is defined as the mass of the skeleton of the clay bounded by the cylinder S and the two planes 0 and z fixed relative to the skeleton and ϵ as the volume of water

saturating unit mass of the porous structure, Eq. (1) is unchanged. If $L(z, t)$ is now defined by the equation

$$L = \int_0^z (\eta + \epsilon) dz,$$

where η is the volume of unit mass of the skeleton, Eq. (2) is also unchanged and Eq. (3) is replaced by

$$\frac{\partial \epsilon}{\partial t} = -\frac{\partial}{\partial z} \left(\frac{K}{\eta + \epsilon} \frac{\partial P}{\partial z} \right). \quad (28)$$

The square root law is now valid if K , P and η can be expressed as functions of ϵ only. Finally, it can be seen that the linearizing approximation, (11), now becomes

$$\frac{K}{\eta + \epsilon} \frac{\partial P}{\partial z} = -k \frac{\partial \epsilon}{\partial z}. \quad (29)$$

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BOOK REVIEWS

Group theory and its application to the quantum mechanics of atomic spectra. By Eugene P. Wigner. Academic Press, New York and London, 1959. xi + 372 pp. \$8.80.

This is an English translation of the original well known German edition (1931) together with the addition of three new chapters. The book still deals entirely with the quantum mechanics of atomic spectra. The translator points out that the translation was motivated by the lack of a good English work on the subject of group theory from "the physicist's point of view."

There is a particularly striking comment by the author in his preface to the effect that von Laue remarked that the most important result of the original edition was the "recognition that almost all rules of spectroscopy follow from the symmetry of the problem."

While this book is written for the mathematical and theoretical physicist it has sections (such as chapter 17) which are written in a descriptive fashion for the purpose of showing the reader how to use results of group theory without becoming any more involved than necessary with the mathematical details. Such sections will certainly appeal to anyone not thoroughly familiar with the subject. At the same time the book should appeal to the advanced student of the subject particularly in view of the recent additions to the work—additions such as the time inversion transformation. The transformation $t \rightarrow -t$ provides an additional symmetry element which transforms a given state into one where all velocities (and spins) are reversed in direction.

The translation seems to have been very worth while. It should make a considerable difference in the number of physicists in the country who learn something about group theory.

ROHN TRUETT

Sampling inspection tables. Second edition. By Harold F. Dodge and Harry G. Romig. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1959. xi + 224 pp. \$8.00.

In this handsome, enlarged, edition of their much used tables the authors have made many new and useful additions. Perhaps the most important is the inclusion of "Operating Characteristic (OC) Curves". These curves give the probability of accepting a lot, when using a chosen sampling scheme, as a function of the percentage of defectives in the lot. Such information is useful in choosing a scheme as it is often important to know how a scheme behaves for a range of percentage defectives.

The introduction to the tables has been expanded so that the procedure and theory of setting up single or double sampling schemes are clearly explained. Two general types of consumer protection are considered: that based on Average Outgoing Quality Limit (AOQL) and that based on Lot Tolerance Percent Defective (LTPD). AOQL schemes ensure that the average quality delivered will, almost invariably, be above some chosen limit. LTPD schemes limit the percent of defectives in each sampled lot. All those concerned with quality control should find this new edition invaluable.

W. FREIBERGER

An introduction to advanced dynamics. By S. W. McCuskey. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959. viii + 263 pp. \$8.50.

In the author's words "The purpose of this introductory book is to familiarize advanced undergraduate students in science and mathematics with a few ideas of classical dynamics not ordinarily treated in their courses in elementary mechanics. . . . The emphasis is on the underlying principles and a few simple and familiar applications for illustrative purposes only."

The end result is quite a readable account of a rather broad range of topics as indicated by the following list of chapter headings: Fundamentals of Newtonian Dynamics, Hamilton's Principle and Lagrange's Equations, Central Force Motion, Dynamics of a Rigid Body, Oscillatory Motion, Hamilton's Equations

(Continued on p. 360)

A DERIVATION OF THE BASIC EQUATIONS FOR HYDRODYNAMIC LUBRICATION WITH A FLUID HAVING CONSTANT PROPERTIES*

BY

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Abstract. In this paper small parameter techniques are used to derive Reynolds' lubrication equations, and refinements thereof, from the full Navier-Stokes equation. An effort has been made to retain rigor in the development comparable to that used in present-day boundary-layer developments.

To derive the differential equations for flow in a curved film of arbitrary thickness requires the use of general tensor analysis. The mathematical manipulations are somewhat involved, but one of the results—a refined Reynolds equation—can be simply written for a journal or slipper bearing as follows:

$$\frac{\partial}{\partial x} \left\{ h^3 \left(1 - \frac{h}{D} \right) \frac{\partial p}{\partial x} \right\} + \frac{\partial}{\partial z} \left\{ h^3 \left(1 + \frac{h}{D} \right) \frac{\partial p}{\partial z} \right\} = 6\mu U \frac{\partial}{\partial x} \left\{ h \left(1 - \frac{h}{3D} \right) \right\}.$$

Here:

- D = shaft diameter (infinite for a slipper bearing)
- h = film thickness
- p = fluid pressure
- U = shaft surface velocity
- x = distance around shaft in direction of rotation
- z = distance parallel to shaft axis
- μ = fluid viscosity

The error of the above differential equation is of the order of $(h/L)^2$, where L is the film length in the direction x .

1. Introduction. The purpose of this paper is to provide a derivation of the basic equations for hydrodynamic lubrication with a fluid having constant properties. It is expected that analytical techniques similar to those employed here can be adapted to the development of equations applicable to fluids having pressure- and temperature-dependent properties.

The derivation given here applies directly to the geometries of both journal and slipper bearings. No difficulties are anticipated in the application of similar analysis to other geometries, such as those which can occur in thrust bearings, but the writer wished to avoid becoming lost in too much generality.

In the literature, a close approach to the present work appears in an article by Wannier [1]. The present work extends that of Wannier in three respects. First, the complete Navier-Stokes equations are used as a starting-point (rather than "Stokes" equations). Second, the mean film surface can have finite curvature. Third, the present approximation procedure is susceptible to improvement without regression.

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Figure 1 shows the physical situation to be analyzed. A shaft rotates within a bearing, which, for the purposes of analysis, is only partial. A circular shaft of radius, R , rotates at angular velocity, ω . The small film thickness, h , between this shaft and the bearing surface can be an arbitrary function of position. It is completely filled with a fluid possessing constant properties, free to flow in and out of the film where it is exposed to ambient conditions at the edge of the bearing.

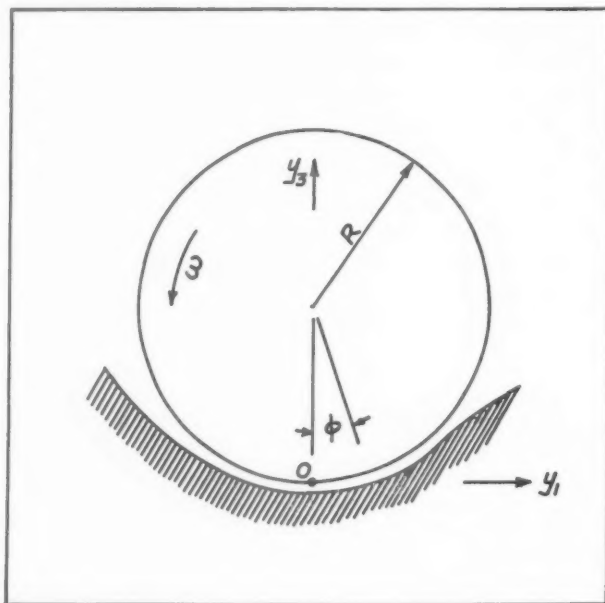


FIG. 1—Schematic diagram of bearing

The problem to be posed here is that of predicting the velocity and pressure distributions within the film. This problem can perhaps best be stated in terms of the variables:

$$\xi^1 \equiv \frac{R}{L} \phi; \quad \xi^2 \equiv \frac{y^2}{L}; \quad \xi^3 \equiv \frac{R - r}{h(\xi^1, \xi^2)}, \quad (1.1)$$

where L is a characteristic dimension of the bearing. The position of a fluid particle is known when its coordinates ξ^1 , ξ^2 and ξ^3 are specified. Likewise, its motion is given by the time derivatives of these quantities, u^i . Thus:

$$u^i \equiv \frac{d\xi^i}{dt}. \quad (1.2)$$

Then it is convenient to define the following set of dimensionless "velocities"

$$u_*^i \equiv \frac{L^2 u^i}{\nu}. \quad (1.3)$$

Corresponding to Fig. 1 we note that:

$$u_*^i = 0; \quad \xi^3 = -1, \quad i = 1, 2, 3.$$

$$u_*^1 = \frac{LR\omega}{\nu}; \quad u_*^i = 0, \quad i = 2, 3; \quad \xi^3 = 0.$$

We can presume that the local film thickness, h , is given by:

$$h = h_0 \exp \{ \varphi(\xi^1, \xi^2) \}, \quad (1.4)$$

where the function φ and its derivatives are $O(1)$. The value of h_0 sets the magnitude, so to speak, of the film thickness throughout the bearing

Finally, we introduce a dimensionless "pressure" π such that

$$\pi \equiv \frac{p}{\rho \left(\frac{\nu}{h_0} \right)^2}. \quad (1.5)$$

The value of π is specified around the periphery of the film, and may there generally be taken as zero.

The significant feature of lubrication hydrodynamics is the smallness of the dimensionless parameter

$$\epsilon \equiv h_0/L. \quad (1.6)$$

Now it is possible to imagine at least two experimental sequences in which this parameter becomes progressively smaller, while at the same time the boundary conditions on the u_*^i and π remain invariant. In the first of these sequences, L is progressively increased with h_0 constant, while R is varied proportionately to L and ω is varied as L^{-2} . Thus u_*^1 remains constant on the shaft, and π on the bearing periphery is unaltered if the external pressure distribution is unaltered. In the second sequence, h_0 is progressively reduced and fluids of equal kinematic viscosity " ν " are employed which have, nevertheless, progressively lower densities; i.e., $\rho \sim h_0^3$. (In the event that the peripheral pressure is constant, as is usual, then this constant pressure can be taken as datum, and the fluid density need not be changed in this second sequence.) Again, the boundary conditions on the u_*^i and π are invariant. The existence of such hypothetical experimental sequences is not necessary to the use of (h_0/L) as a small parameter in a mathematical expansion, but it does tend to assure that there will be a range of experimental conditions for which the early terms of the expansion will provide an adequate description of the observed variables.

Although the boundary conditions of the problem can be made independent of ϵ , as shown above, the differential equations for the u_*^i and π in terms of ξ^1, ξ^2, ξ^3 , do contain ϵ in such a manner that a non-singular perturbation problem can be formulated. Thus, we hypothesize that we can represent the dependent variables as follows:

$$u_*^i = U_0^i(\xi^1, \xi^2, \xi^3) + \epsilon U_1^i(\xi^1, \xi^2, \xi^3) + \epsilon^2 U_2^i(\xi^1, \xi^2, \xi^3) \text{ etc.}$$

$$\pi = P_0(\xi^1, \xi^2, \xi^3) + \epsilon P_1(\xi^1, \xi^2, \xi^3) + \epsilon^2 P_2(\xi^1, \xi^2, \xi^3). \quad (1.7)$$

When these series are substituted into the complete Navier-Stokes equations, and the coefficients of the various powers of ϵ are equated to zero, a sequence of differential equations is generated for the U_k^i and P_k . The lowest order equations are the original

equations of Reynolds. Successively better solutions can be found by solving the higher-order differential equations in a manner explicitly indicated in this paper. If the series in ϵ [Eq. (1.7)] converge, true solutions of the Navier-Stokes equation should result.

2. The metric tensor. The simplest manner of obtaining the differential equations for the u_*^i and π is through the use of tensor calculus. Hence we first compute the covariant components of the metric tensor. They are given by:

$$g_{\alpha\beta} = \sum_{i=1}^{i=3} \frac{\partial y^i}{\partial \xi^\alpha} \frac{\partial y^i}{\partial \xi^\beta}. \quad (2.1)$$

The intermediate algebraic details required to obtain the $g_{\alpha\beta}$ are recorded in the Appendix. The resulting matrix is given below

$$\frac{g_{\alpha\beta}}{L^2} = G_{\alpha\beta} = \begin{vmatrix} \left(1 - \xi^3 \frac{h}{R}\right)^2 + \left(\xi^3 \frac{\partial h}{L \partial \xi^1}\right)^2 & \left(\xi^3 \frac{\partial h}{L \partial \xi^1}\right)^2 \frac{\partial h}{\partial \xi^1} \frac{\partial h}{\partial \xi^2} & \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^1} \\ \left(\xi^3 \frac{\partial h}{L \partial \xi^1}\right)^2 \frac{\partial h}{\partial \xi^1} \frac{\partial h}{\partial \xi^2} & 1 + \left(\xi^3 \frac{\partial h}{L \partial \xi^2}\right)^2 & \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^2} \\ \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^1} & \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^2} & \left(\frac{h}{L}\right)^2 \end{vmatrix}. \quad (2.2)$$

The determinant of the matrix $|g_{\alpha\beta}|$ is:

$$g = |g_{\alpha\beta}| = \left(1 - \xi^3 \frac{h}{R}\right)^2 L^4 h^2. \quad (2.3)$$

The contravariant components of the metric tensor are found from the formula

$$g^{\alpha\beta} = \frac{1}{g} \cdot \text{Cofactor of } g_{\beta\alpha} \text{ in } g. \quad (2.4)$$

As shown in the Appendix, the contravariant array is as follows

$$L^2 g^{\alpha\beta} = G^{\alpha\beta} = \begin{vmatrix} \frac{1}{\left(1 - \xi^3 \frac{h}{R}\right)^2} & 0 & \frac{\xi^3 \frac{\partial \ln h}{\partial \xi^1}}{\left(1 - \xi^3 \frac{h}{R}\right)^2} \\ 0 & 1 & -\xi^3 \frac{\partial \ln h}{\partial \xi^2} \\ \frac{-\xi^3 \frac{\partial \ln h}{\partial \xi^1}}{\left(1 - \xi^3 \frac{h}{R}\right)^2} & -\xi^3 \frac{\partial \ln h}{\partial \xi^2} \frac{L^2}{h^2} + \frac{\left(\xi^3 \frac{\partial \ln h}{\partial \xi^1}\right)^2}{\left(1 - \xi^3 \frac{h}{R}\right)^2} + \left(\xi^3 \frac{\partial \ln h}{\partial \xi^2}\right)^2 & \end{vmatrix}. \quad (2.5)$$

Finally, the Euclidean Christoffel symbols are required. They are given by the formula:

$$\Gamma_{\alpha\beta}^i(\xi^1, \xi^2, \xi^3) = \frac{1}{2} g^{i\sigma} \left(\frac{\partial g_{\sigma\beta}}{\partial \xi^\alpha} + \frac{\partial g_{\sigma\alpha}}{\partial \xi^\beta} - \frac{\partial g_{\alpha\beta}}{\partial \xi^\sigma} \right) \quad (2.6)$$

or:

$$\Gamma_{\alpha\beta}^i(\xi^1, \xi^2, \xi^3) = \frac{1}{2} G^{i\sigma} \left(\frac{\partial G_{\sigma\beta}}{\partial \xi^\alpha} + \frac{\partial G_{\sigma\alpha}}{\partial \xi^\beta} - \frac{\partial G_{\alpha\beta}}{\partial \xi^\sigma} \right). \quad (2.7)$$

Explicit expressions for the $\Gamma_{\alpha\beta}^i$ will not be given; rather we shall determine their orders-of-magnitude in terms of ϵ .

We first note that all $G^{\alpha\beta}$ are $O(\epsilon^0)$, with the exception of G^{33} , which is $O(\epsilon^{-2})$. Thus:

$$G^{\alpha\beta} = O\{\exp(-2\delta_3^\alpha \delta_3^\beta \ln \epsilon)\}. \quad (2.8)$$

Next we note that all derivatives of the $G_{\alpha\beta}$ are $O(\epsilon^2)$, with the exception of the derivatives of G_{11} , which are $O(\epsilon)$. Thus:

$$\frac{\partial G_{\alpha\beta}}{\partial \xi^k} = O\{\exp(2 - \delta_\alpha^1 \delta_\beta^1 \ln \epsilon)\}. \quad (2.9)$$

Substituting Eqs. (2.8) and (2.9) into Eq. (2.7), we obtain:

$$\begin{aligned} \Gamma_{\alpha\beta}^i = O\{\exp[(-2\delta_3^i \delta_3^\sigma + 2 - \delta_\sigma^1 \delta_\beta^1) \ln \epsilon]\} \\ + O\{\exp[(-2\delta_3^i \delta_3^\sigma + 2 - \delta_\alpha^1 \delta_\sigma^1) \ln \epsilon]\} \\ + O\{\exp[(-2\delta_3^i \delta_3^\sigma + 2 - \delta_\alpha^1 \delta_\beta^1) \ln \epsilon]\} \quad \sigma = 1, 2, 3 \end{aligned} \quad (2.10)$$

By picking the lowest power of ϵ for each $\Gamma_{\alpha\beta}^i$, we reach the following conclusions.

$$\left. \begin{aligned} i \neq 3, \quad \alpha \neq 1, \quad \beta \neq 1: \quad \Gamma_{\alpha\beta}^i &= O(\epsilon^2) \\ i \neq 3, \quad \alpha = 1, \quad \text{or} \quad \beta = 1, \quad \text{or} \quad \alpha = \beta = 1: \quad \Gamma_{\alpha\beta}^i &= O(\epsilon) \\ i = 3, \quad \alpha \neq 1, \quad \text{or} \quad \beta \neq 1: \quad \Gamma_{\alpha\beta}^i &= O(\epsilon^0) \\ i = 3, \quad \alpha = \beta = 1: \quad \Gamma_{\alpha\beta}^i &= O(\epsilon^{-1}). \end{aligned} \right\} \quad (2.11)$$

3. Reduction of the momentum and mass continuity equations. The Navier-Stokes equations for a fluid having constant properties are [2]:

$$\begin{aligned} 0 = \nu g^{\alpha\beta} \left[\frac{\partial^2 u^i}{\partial \xi^\alpha \partial \xi^\beta} + \Gamma_{\sigma\beta}^i \frac{\partial u^\sigma}{\partial \xi^\alpha} + \Gamma_{\sigma\alpha}^i \frac{\partial u^\sigma}{\partial \xi^\beta} - \Gamma_{\alpha\beta}^\sigma \frac{\partial u^i}{\partial \xi^\sigma} + \left(\frac{\partial \Gamma_{\sigma\alpha}^i}{\partial \xi^\beta} + \Gamma_{\tau\beta}^i \Gamma_{\sigma\alpha}^\tau - \Gamma_{\sigma\tau}^i \Gamma_{\alpha\beta}^\tau \right) u^\sigma \right] \\ - u^\alpha \left(\frac{\partial u^i}{\partial \xi^\alpha} + \Gamma_{\sigma\alpha}^i u^\sigma \right) - \frac{1}{\rho} g^{i\alpha} \frac{\partial p}{\partial \xi^\alpha}. \end{aligned} \quad (3.1)$$

In terms of the dimensionless velocities u_*^i and the dimensionless pressure, π , this equation becomes:

$$\begin{aligned} 0 = G^{\alpha\beta} \left[\frac{\partial^2 u_*^i}{\partial \xi^\alpha \partial \xi^\beta} + \Gamma_{\sigma\beta}^i \frac{\partial u_*^\sigma}{\partial \xi^\alpha} + \Gamma_{\sigma\alpha}^i \frac{\partial u_*^\sigma}{\partial \xi^\beta} - \Gamma_{\alpha\beta}^\sigma \frac{\partial u_*^i}{\partial \xi^\sigma} \right. \\ \left. + \left(\frac{\partial \Gamma_{\sigma\alpha}^i}{\partial \xi^\beta} + \Gamma_{\tau\beta}^i \Gamma_{\sigma\alpha}^\tau - \Gamma_{\sigma\tau}^i \Gamma_{\alpha\beta}^\tau \right) u_*^\sigma \right] \\ - u_*^\alpha \left(\frac{\partial u_*^i}{\partial \xi^\alpha} + \Gamma_{\sigma\alpha}^i u_*^\sigma \right) - \epsilon^{-2} G^{i\alpha} \frac{\partial \pi}{\partial \xi^\alpha}. \end{aligned} \quad (3.2)$$

With $i \neq 3$, we retain in Eq. (3.2) those terms of lowest power in ϵ ; i.e., of $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$. The resulting, simplified equation is:

$$\begin{aligned} 0 = -G^{11} \Gamma_{11}^3 \frac{\partial u_*^3}{\partial \xi^3} + G^{33} \left[\frac{\partial^2 u_*^3}{(\partial \xi^3)^2} + 2\Gamma_{13}^3 \frac{\partial u_*^1}{\partial \xi^3} + \Gamma_{33}^3 \frac{\partial u_*^3}{\partial \xi^3} + \frac{\partial \Gamma_{13}^3}{\partial \xi^3} u_*^1 \right] \\ - \epsilon^{-2} G^{i\alpha} \frac{\partial \pi}{\partial \xi^\alpha}. \end{aligned} \quad (3.3)$$

Explicit expressions for the G^{ab} are given by (2.5). The required $\Gamma_{\sigma\tau}^a$ in Eq. (3.3) are given with sufficient accuracy as:

$$\Gamma_{13}^i = \frac{1}{2}G^{i1} \frac{\partial G_{11}}{\partial \xi^3} = \frac{1}{2}G^{i1} \left(-2 \frac{h}{R}\right) = -G^{i1} \frac{h}{R} = 0(\epsilon), \quad (3.4)$$

$$\frac{\partial \Gamma_{13}^i}{\partial \xi^3} = 0(\epsilon^2) \quad \text{because} \quad \frac{\partial h}{\partial \xi^3} = 0. \quad (3.5)$$

$$\Gamma_{33}^3 = \frac{1}{2}G^{3\sigma} \left[2 \frac{\partial G_{\sigma 3}}{\partial \xi^3} - \frac{\partial G_{33}}{\partial \xi^\sigma} \right] = 0(\epsilon^2) \quad \text{because} \quad \frac{\partial G_{33}}{\partial \xi^3} = 0 \quad (3.6)$$

$$\Gamma_{11}^3 = -\frac{1}{2}G^{33} \frac{\partial G_{11}}{\partial \xi^3} = \frac{h}{R} \left(\frac{L}{h}\right)^2 = \frac{L}{R} \left(\frac{L}{h}\right) = 0\left(\frac{1}{\epsilon}\right). \quad (3.7)$$

By virtue of these relations, Eq. (3.3) becomes:

$$0 = \left(\frac{L}{h}\right)^2 \left[\frac{\partial^2 u_*^i}{(\partial \xi^3)^2} - 2G^{i1} \frac{h}{R} \frac{\partial u_*^1}{\partial \xi^3} - \frac{h}{R} \frac{\partial u_*^i}{\partial \xi^3} \right] - \epsilon^{-2} \left[G^{i1} \frac{\partial \pi}{\partial \xi^1} + G^{i2} \frac{\partial \pi}{\partial \xi^2} + G^{i3} \frac{\partial \pi}{\partial \xi^3} \right]. \quad (3.8)$$

With $i = 3$, we again seek the two lowest powers of ϵ in Eq. (3.2). It is readily found that, correct in terms of $0(\epsilon^{-4})$ and $0(\epsilon^{-3})$,

$$\frac{\partial \pi}{\partial \xi^3} = 0. \quad (3.9)$$

Now we examine the mass-continuity equation. In tensor form, it can be written as:

$$\frac{\partial}{\partial \xi^\alpha} (g^{\frac{1}{2}} u^\alpha) = 0, \quad (3.10)$$

where, of course,

$$g = \left(1 - \xi^3 \frac{h}{R}\right)^2 L^4 h^2. \quad (3.11)$$

Since u_*^3 vanishes at $\xi_3 = 0$ and $\xi_3 = -1$, we can integrate Eq. (3.10) to get:

$$\frac{\partial}{\partial \xi^1} \int_0^{-1} \left(1 - \xi^3 \frac{h}{R}\right) h u_*^1 d\xi^3 + \frac{\partial}{\partial \xi^2} \int_0^{-1} \left(1 - \xi^3 \frac{h}{R}\right) h u_*^2 d\xi^3 = 0. \quad (3.12)$$

This last equation is exact.

4. Reynolds equation with first correction terms. In this section we shall obtain a Reynolds equation valid to $0(h_0/L)$. From Eqs. (3.8) we obtain:

$$\frac{\partial^2 u_*^1}{(\partial \xi^3)^2} - 2G^{11} \frac{h}{R} \frac{\partial u_*^1}{\partial \xi^3} - \frac{h}{R} \frac{\partial u_*^1}{\partial \xi^3} = G^{11} \left(\frac{h}{h_0}\right)^2 \frac{\partial \pi}{\partial \xi^1}, \quad (4.1)$$

$$\frac{\partial^2 u_*^2}{(\partial \xi^3)^2} - \frac{h}{R} \frac{\partial u_*^2}{\partial \xi^3} = G^{22} \left(\frac{h}{h_0}\right)^2 \frac{\partial \pi}{\partial \xi^2}, \quad (4.2)$$

where:

$$G^{11} = \frac{1}{\left(1 - \xi^3 \frac{h}{R}\right)^2} = 1 + 2\xi^3 \frac{h}{R} + \dots \quad (4.3)$$

$$G^{22} = 1. \quad (4.4)$$

Retention of terms through $O(\epsilon)$ in Eqs. (4.1) and (4.2) gives:

$$\frac{\partial^2 u_*^1}{\partial (\xi^3)^2} - 3 \frac{h}{R} \frac{\partial u_*^1}{\partial \xi^3} = \left(1 + 2\xi^3 \frac{h}{R}\right) \left(\frac{h}{h_0}\right)^2 \frac{\partial \pi}{\partial \xi^1} \quad (4.5)$$

and

$$\frac{\partial^2 u_*^2}{\partial (\xi^3)^2} - \frac{h}{R} \frac{\partial u_*^2}{\partial \xi^3} = \left(\frac{h}{h_0}\right)^2 \frac{\partial \pi}{\partial \xi^2}. \quad (4.6)$$

Because, to the degree of approximation presently being retained, π is independent of ξ^3 , Eqs. (4.5) and (4.6) can be integrated with respect to ξ^3 . For u_*^1 we obtain:

$$u_*^1 = \left(\frac{LR\omega}{\nu} + \frac{q_1 \xi^3}{2}\right)(1 + \xi^3) + \frac{3h}{R} \left[\xi^3 \left\{ \frac{LR\omega}{2\nu} - \frac{q_1}{36} \right\} + (\xi^3)^2 \left\{ \frac{LR\omega}{2\nu} + \frac{q_1}{4} \right\} + (\xi^3)^3 \frac{5q_1}{18} \right], \quad (4.7)$$

where

$$q_1 \equiv \left(\frac{h}{h_0}\right)^2 \frac{\partial \pi}{\partial \xi^1}$$

and terms of $O(\epsilon^2)$ and higher have been neglected.

For u_*^2 we obtain:

$$u_*^2 = q_2 \left[\frac{\xi^3(1 + \xi^3)}{2} + \frac{h}{R} \left\{ \frac{\xi^3}{12} + \frac{(\xi^3)^2}{4} + \frac{(\xi^3)^3}{6} \right\} \right], \quad (4.8)$$

where

$$q_2 \equiv \left(\frac{h}{h_0}\right)^2 \frac{\partial \pi}{\partial \xi^2}.$$

The "velocities" u_*^1 and u_*^2 from the above Eqs. (4.7) and (4.8) can now be substituted into Eq. (3.12) and the definite integrations called for in this last equation can be carried out explicitly. When terms through $O(\epsilon)$ are retained, the result is:

$$\begin{aligned} \frac{\partial}{\partial \xi^1} \left(\frac{h}{h_0} \right) \left[\left(\frac{h}{h_0} \right)^2 \left(1 - \frac{h}{2R} \right) \frac{\partial \pi}{\partial \xi^1} - \left(6 - \frac{h}{R} \right) \frac{LR\omega}{\nu} \right] \\ + \frac{\partial}{\partial \xi^2} \left(\frac{h}{h_0} \right) \left[\left(\frac{h}{h_0} \right)^2 \left(1 + \frac{h}{2R} \right) \frac{\partial \pi}{\partial \xi^2} \right] = 0. \end{aligned} \quad (4.9)$$

In more conventional variables, this equation becomes:

$$\frac{\partial}{\partial x} \left\{ h^3 \left(1 - \frac{h}{D} \right) \frac{\partial p}{\partial x} \right\} + \frac{\partial}{\partial z} \left\{ h^3 \left(1 + \frac{h}{D} \right) \frac{\partial p}{\partial z} \right\} = 6\mu U \frac{\partial}{\partial x} \left\{ h \left(1 - \frac{h}{3D} \right) \right\}, \quad (4.10)$$

where D is the shaft diameter, x is distance around the shaft, and z is distance along its axis.

In all usual applications, $h/D \ll 1$, so that Eq. (4.10) is essentially Reynolds' Lubrication Equation with correction terms. It has been derived in this paper from the full Navier-Stokes equation by a systematic procedure employing $\epsilon \equiv h_0/L$ as a small parameter. Now:

$$\frac{h}{D} = \frac{h}{h_0} \cdot \frac{h_0}{L} \cdot \frac{L}{D} = \epsilon \left(\frac{h}{h_0} \right) \left(\frac{L}{D} \right). \quad (4.11)$$

When the characteristic length L is maintained constant and D is taken progressively larger, $h/D \rightarrow 0$, and the equation for a plane slipper bearing is obtained. It is seen that Reynolds' equation for a plane slipper bearing has an error of $O(\epsilon^2)$, rather than $O(\epsilon)$.

5. Development of solutions to the Navier-Stokes equations. The general approximation procedure, of which we have just examined the first few steps, will now be outlined.

Substitution of the series 1.7 into Eq. (3.8) gives the following sets of equations which result from equating to zero the coefficients of the successive powers of " ϵ "

$$\frac{\partial^2 U_k^i}{\partial (\xi^3)^2} = \left(\frac{h}{h_0} \right)^2 \frac{\partial P_k}{\partial \xi^1} + H_{k-1}(\xi^1, \xi^2, \xi^3), \quad (5.1)$$

$$\frac{\partial^2 U_k^2}{\partial (\xi^3)^2} = \left(\frac{h}{h_0} \right)^2 \frac{\partial P_k}{\partial \xi^2} + I_{k-1}(\xi^1, \xi^2, \xi^3), \quad (5.2)$$

$$\frac{\partial P_k}{\partial \xi^3} = K_{k-1}(\xi^1, \xi^2, \xi^3). \quad (5.3)$$

In these equations, the functions H_{k-1} , I_{k-1} and K_{k-1} are known functions of space involving velocity and pressure functions of lower order obtained earlier in the approximation procedure. For example, $H_{k-1}(\xi^1, \xi^2, \xi^3)$ would involve

$$U_{k-1}^1, U_{k-1}^2, U_{k-1}^3; \quad U_{k-2}^1, U_{k-2}^2, \text{ etc.}$$

$$P_{k-1}, P_{k-2}, \text{ etc.}$$

and their derivatives.

Now from Eq. (5.3) we have:

$$P_k(\xi^1, \xi^2, \xi^3) = P_k(\xi^1, \xi^2, 0) + \int_0^{\xi^3} \frac{\partial P_k}{\partial \xi^3} d\xi^3. \quad (5.4)$$

Or:

$$\frac{\partial P_k}{\partial \xi^1} = \frac{\partial P_k(0)}{\partial \xi^1} + L_{k-1}(\xi^1, \xi^2, \xi^3). \quad (5.5)$$

In this equation, and hereafter, we shall abbreviate

$$P_k(0) \equiv P_k(\xi^1, \xi^2, 0).$$

Substitution of Eq. (5.5) into Eq. (5.1) gives:

$$\frac{\partial^2 U_k^1}{\partial (\xi^3)^2} = \left(\frac{h}{h_0} \right)^2 \frac{\partial P_k(0)}{\partial \xi^1} + M_{k-1}(\xi^1, \xi^2, \xi^3). \quad (5.6)$$

This last equation can be integrated twice with respect to ξ^3 . For $k > 0$, the boundary conditions are:

$$U_k^i(\xi^1, \xi^2, 0) = U_k^i(\xi^1, \xi^2, -1) = 0. \quad (5.7)$$

After integration we have:

$$U_k^1(\xi^1, \xi^2, \xi^3) = \left(\frac{h}{h_0} \right)^2 \frac{\partial P_k(0)}{\partial \xi^1} \frac{(\xi^3)^2}{2} + N_{k-1}(\xi^1, \xi^2, \xi^3). \quad (5.8)$$

A like equation is obtained for U_k^2 . Thus:

$$U_k^2(\xi^1, \xi^2, \xi^3) = \left(\frac{h}{h_0}\right)^2 \frac{\partial P_k(0)}{\partial \xi^2} \frac{(\xi^3)^2}{2} + O_{k-1}(\xi^1, \xi^2, \xi^3). \quad (5.9)$$

The mass-continuity equation, Eq. (3.12), gives the following recursion equation involving the U_k^i .

$$\frac{\partial}{\partial \xi^1} \int_0^{-1} h U_k^1(\xi^1, \xi^2, \xi^3) d\xi^3 + \frac{\partial}{\partial \xi^2} \int_0^{-1} h U_k^2(\xi^1, \xi^2, \xi^3) d\xi^3 = Q_{k-1}(\xi^1, \xi^2). \quad (5.10)$$

Substitution of the U_k^1 and U_k^2 from Eqs. (5.8) and (5.9) gives:

$$\frac{\partial}{\partial \xi^1} \left(\frac{h}{h_0}\right)^3 \frac{\partial P_k(0)}{\partial \xi^1} + \frac{\partial}{\partial \xi^2} \left(\frac{h}{h_0}\right)^3 \frac{\partial P_k(0)}{\partial \xi^2} = R_{k-1}(\xi^1, \xi^2). \quad (5.11)$$

Here we have a partial differential equation for $P_k(0)$ as a function of ξ^1 and ξ^2 . It is the "diffusion equation" with a variable "diffusion coefficient" $(h/h_0)^3$ and a source term, R_{k-1} . For $k > 0$ the boundary condition to be imposed at the edge of the fluid film is $P_k(0) = 0$.

In principle, Eq. (5.11) can be solved. Then $P_k(\xi^1, \xi^2, \xi^3)$ can be found from Eq. (5.4) and $U_k^1(\xi^1, \xi^2, \xi^3)$ and $U_k^2(\xi^1, \xi^2, \xi^3)$ from Eqs. (5.8) and (5.9). All information for repeating the same process for U_{k+1}^1 , U_{k+1}^2 and P_{k+1} is now available. Provided that the series (1.7) converge, solutions are developed for the complete Navier-Stokes equations subject to the boundary conditions

$$u_*^1 = U_0^1 = \frac{LR\omega}{\nu}; \quad \xi^3 = 0$$

$$\pi = P_0 = f(t),$$

where the edge of the fluid film is described by the parametric equations

$$\xi^1 = \xi^1(t); \quad \xi^2 = \xi^2(t).$$

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2. A. D. Michal, *Matrix and tensor calculus*, John Wiley and Sons, New York, 1947

Appendix. Calculation of covariant components of metric tensor. The transformation inverse to Eqs. (1.1) is:

$$y^1 = r \sin \varphi = (R - \xi^3 h) \sin \varphi = (R - \xi^3 h) \sin \left(\frac{L\xi^1}{R}\right), \quad (A.1)$$

$$y^2 = L\xi^2, \quad (A.2)$$

$$y^3 = R - r \cos \varphi = R - (R - \xi^3 h) \cos \left(\frac{L\xi^1}{R}\right). \quad (A.3)$$

From Eqs. (A.1-A.3):

$$\frac{\partial y^1}{\partial \xi^1} = \frac{L}{R} (R - \xi^3 h) \cos \varphi - \xi^3 \frac{\partial h}{\partial \xi^1} \sin \varphi, \quad (\text{A.4})$$

$$\frac{\partial y^1}{\partial \xi^3} = -\xi^3 \frac{\partial h}{\partial \xi^2} \sin \varphi, \quad (\text{A.5})$$

$$\frac{\partial y^1}{\partial \xi^2} = -h \sin \varphi, \quad (\text{A.6})$$

$$\frac{\partial y^2}{\partial \xi^1} = 0; \quad \frac{\partial y^2}{\partial \xi^2} = L; \quad \frac{\partial y^2}{\partial \xi^3} = 0, \quad (\text{A.7})$$

$$\frac{\partial y^3}{\partial \xi^1} = \frac{L}{R} (R - \xi^3 h) \sin \varphi + \xi^3 \frac{\partial h}{\partial \xi^1} \cos \varphi, \quad (\text{A.8})$$

$$\frac{\partial y^3}{\partial \xi^2} = \xi^3 \frac{\partial h}{\partial \xi^2} \cos \varphi, \quad (\text{A.9})$$

$$\frac{\partial y^3}{\partial \xi^3} = h \cos \varphi. \quad (\text{A.10})$$

From the foregoing derivatives the covariant components $g_{\alpha\beta}$ can be calculated according to Eq. (2.1). For example:

$$\begin{aligned} g_{11} = & \left\{ \frac{L}{R} (R - \xi^3 h) \right\}^2 \cos^2 \varphi + \left(\xi^3 \frac{\partial h}{\partial \xi^1} \right)^2 \sin^2 \varphi - 2 \frac{L}{R} (R - \xi^3 h) \xi^3 \frac{\partial h}{\partial \xi^1} \sin \varphi \cos \varphi \\ & + \left\{ \frac{L}{R} (R - \xi^3 h) \right\}^2 \sin^2 \varphi + \left(\xi^3 \frac{\partial h}{\partial \xi^1} \right)^2 \cos^2 \varphi + 2 \frac{L}{R} (R - \xi^3 h) \xi^3 \frac{\partial h}{\partial \xi^1} \sin \varphi \cos \varphi \end{aligned} \quad (\text{A.11})$$

or:

$$g_{11} = \left\{ \frac{L}{R} (R - \xi^3 h) \right\}^2 + \left(\xi^3 \frac{\partial h}{\partial \xi^1} \right)^2. \quad (\text{A.12})$$

Next:

$$g_{22} = \left(\xi^3 \frac{\partial h}{\partial \xi^2} \right)^2 \sin^2 \varphi + L^2 + \left(\xi^3 \frac{\partial h}{\partial \xi^2} \right)^2 \cos^2 \varphi = L^2 + \left(\xi^3 \frac{\partial h}{\partial \xi^2} \right)^2. \quad (\text{A.13})$$

The other $g_{\alpha\beta}$ can be found by similar summations. All are combined in the matrix appearing as Eq. (2.2). The third-order determinant of this matrix is readily found to be

$$|g_{\alpha\beta}| = \left(1 - \xi^3 \frac{h}{R} \right) L^4 h^2. \quad (\text{A.14})$$

Calculation of contravariant components of metric tensor. Equation (2.4) is used to calculate the contravariant components, $g^{\alpha\beta}$. For example, g^{33} is found as follows:

$$g^{33} = (-1)^{3+3} L^4 \left[\left(1 - \xi^3 \frac{h}{R} \right)^2 L^4 h^2 \right]^{-1} \begin{vmatrix} \left(1 - \xi^3 \frac{h}{R} \right)^2 + \left(\frac{\xi^3}{L} \frac{\partial h}{\partial \xi^1} \right)^2 & \left(\frac{\xi^3}{L} \right)^2 \frac{\partial h}{\partial \xi^1} \frac{\partial h}{\partial \xi^2} \\ \left(\frac{\xi^3}{L} \right)^2 \frac{\partial h}{\partial \xi^1} \frac{\partial h}{\partial \xi^2} & 1 + \left(\frac{\xi^3}{L} \frac{\partial h}{\partial \xi^2} \right)^2 \end{vmatrix} \quad (\text{A.15})$$

or:

$$g^{33} = \frac{1}{h^2} + \frac{1}{L^2 \left(1 - \xi^3 \frac{h}{R}\right)^2} \left(\xi^3 \frac{\partial \ln h}{\partial \xi^1} \right)^2 + \frac{1}{L^2} \left(\xi^3 \frac{\partial \ln h}{\partial \xi^2} \right)^2. \quad (\text{A.16})$$

Likewise, we find that:

$$g^{23} = (-1)^{2+3} L^4 \left[\left(1 - \xi^3 \frac{h}{R}\right)^2 L^4 h^2 \right]^{-1} \left| \begin{array}{cc} \left(1 - \xi^3 \frac{h}{R}\right)^2 + \left(\frac{\xi^3}{L} \frac{\partial h}{\partial \xi^1}\right)^2 & \left(\frac{\xi^3}{L}\right)^2 \frac{\partial h}{\partial \xi^1} \frac{\partial h}{\partial \xi^2} \\ \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^1} & \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^2} \end{array} \right| \quad (\text{A.17})$$

or:

$$g^{23} = -\frac{\xi^3}{L^2} \frac{\partial \ln h}{\partial \xi^2}. \quad (\text{A.18})$$

The matrix in Eq. (2.5) gives the remaining components.

BOOK REVIEWS

(Continued from p. 343)

and Phase Space, The Hamilton Jacobi Equation. However, the main purpose of the book, i.e. to place the emphasis on the underlying principles, cannot be regarded as fulfilled. In fact the underlying principles receive a rather cursory treatment and the presentation of some of them lacks clarity and precision and may be misleading. It may suffice to cite one instance. It is stated on p. 44: "Suppose there are n particles in the system and that \vec{F}_i represents the resultant force acting on the i th particle. Then the principle of virtual work states that the system will be in equilibrium when

$$\sum_{i=1}^n \vec{F}_i \cdot \delta \vec{r}_i = 0. \quad (2-5)$$

Here $\delta \vec{r}_i$ is the virtual displacement of the i th particle and is arbitrary except for the constraint conditions." Of course, if \vec{F}_i is indeed the resultant force, then $\delta \vec{r}_i$ need not be compatible with constraints. By \vec{F}_i the author undoubtedly meant the resultant applied force. But nowhere in the book are the applied forces distinguished from the forces of constraint. In fact, constraint forces such as string tensions, surface supports are listed as applied forces on p. 48 of the book.

E. T. ONAT

Mathematics in physics and engineering. By J. Irving and N. Mullineux. Academic Press, New York and London, 1959. xvii + 883 pp. \$11.50.

The following table of contents indicates the scope of the work:

1. Introduction to partial differential equations (68 pp.).
2. Ordinary differential equations: Frobenius' and other methods of solution (58 pp.).
3. Bessel and Legendre functions (80 pp.).
4. The Laplace and other transforms (45 pp.).
5. Matrices (59 pp.).
6. Analytical methods in classical and wave mechanics (55 pp.).
7. Calculus of variations (70 pp.).
8. Complex variable theory and conformal transformations (64 pp.).
9. The calculus of residues (82 pp.).
10. Transform theory (73 pp.).
11. Numerical methods (65 pp.).
12. Integral equations (56 pp.).

There is an appendix (84 pp.) of miscellaneous elementary topics in pure mathematics which supplements the text.

The work is suitable as a basis for a first year graduate course on mathematical methods for physicists and engineers. It is clearly and concisely written, exceptionally well printed and contains a wealth of worked and unworked examples with solutions taken from many areas of applied mathematics; the emphasis in both text and the examples is not on mathematical niceties but on applications to, for instance, elasticity, supersonic flow, electromagnetism, wave mechanics and heat flow. There is a useful list of general references at the end of each chapter.

WALTER FREIBERGER

Fachbegriffe der Programmierungstechnik. Edited by J. Heinhold. R. Oldenbourg Verlag, Munchen, 1959. 34 pp. \$1.05.

There has been no field in recent years in which questions of terminology have been as confusing as in digital computer science. The present booklet is published by the Gesellschaft für Angewandte

(Continued on p. 374)

AN IMPLICIT, NUMERICAL METHOD FOR SOLVING THE TWO-DIMENSIONAL HEAT EQUATION*

BY

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1. Introduction. We develop an implicit scheme for the numerical solution of the two-dimensional heat-flow problem. In the linear case we are able to solve exactly the full two-dimensional set of implicit equations. This solution is possible because we choose a difference scheme for which the equations are factorable into two one-dimensional sets. This factorability is basic to the method.

We extend this method to non-linear equations and non-rectangular regions by the use of an iterative scheme to solve the implicit equations obtained. This scheme provides second-order convergence, and in the cases we have tested only a very few iterations per time step were required.

The method is proved to be unconditionally stable both in the linear and non-linear cases. (We consider only a special set of non-linear problems in the stability analysis.) We prove stability in the linear case by the usual type of Fourier analysis and superposition of solutions. In the non-linear case we show that the norms of the solutions of the difference equation with homogeneous boundary conditions tend to zero as time tends to infinity. This method is limited in mesh size and time-step length only by the requirements of accuracy and not stability.

By way of illustration we include a discussion of two numerical examples.

2. The linear case. The basic partial differential equation which we wish to consider is

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \alpha^{-2} \frac{\partial \theta}{\partial t}, \quad (2.1)$$

where x and y are space coordinates, t is time, and α is a constant. We wish to approximate this differential equation by a finite difference scheme to allow the approximate numerical calculation of the function θ . If we denote spatial points by either the pair of indices (ij) or the pair (kl) , and time by n , then we may write, using the Einstein summation convention, a general, linear difference scheme as

$$B_{kl}^{ij} {}^n \theta_{ij} = {}^n a_{kl}, \quad (2.2)$$

where the ${}^n a_{kl}$ depend on quantities with time earlier than n . We wish to choose a particular B_{kl}^{ij} which will both represent (2.1) and allow for easy calculation of ${}^n \theta_{kl}$ for successive values n . Clearly, an explicit differencing scheme ($B_{ij}^{kl} \propto \delta_i^k \delta_j^l$, where δ_i^k is the Kronecker delta) has both these properties, however, such a scheme is well known to have the disadvantage of being only conditionally stable. Implicit schemes are usually

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unconditionally stable. One such implicit scheme is the Douglas-Peaceman alternating-direction method [1]. It represents the application of the one-dimensional Bruce, Peaceman, Rachford, and Rice [2] method to two dimensions. It has the disadvantage that only one direction is treated exactly at each advance in time. We feel it desirable to choose a differencing scheme which will permit an exact treatment over the entire two dimensional mesh. With this view in mind we note that if we may factor B_{kl}^{ii} as

$$B_{kl}^{ii} = A_k^i B_l^i \quad (2.3)$$

then we may triangularly resolve A_k^i and B_l^i separately as was done in one dimension by Bruce, Peaceman, Rachford, and Rice and obtain a method for the direct calculation of the ${}^n\theta_{kl}$ from ${}^m\theta_{kl}$ with $m < n$. We shall now restrict ourselves to 9-point difference schemes, i.e.,

$$B_{kl}^{ii} = 0 \quad \text{if} \quad |i - k| > 1, \quad \text{or} \quad |j - l| > 1. \quad (2.4)$$

It will be shown later that 5-point difference schemes are unfactorable. Thus let A_k^i and B_l^i be triple-diagonal matrices,

$$\begin{aligned} A_k^i &= (x_1 \delta_k^{i+1} + x_2 \delta_k^i + x_3 \delta_k^{i-1}), \\ B_l^i &= (y_1 \delta_l^{i+1} + y_2 \delta_l^i + y_3 \delta_l^{i-1}). \end{aligned} \quad (2.5)$$

In order to represent Eq. (2.1), B_{kl}^{ii} must be a numerical representation of the Laplacian plus whatever part of the representation of the time derivative involves θ at the advanced time. Thus (suppressing the superscript n),

$$B_{kl}^{ii} \theta_{ii} \text{ represents } \beta \theta_{kl} + \nabla^2 \theta_{kl}. \quad (2.6)$$

From the symmetry properties of the Laplacian, we may restrict the values of B_{kl}^{ii} (allowing a different mesh-spacing in the x and y directions) to

$$\begin{aligned} B_{kl}^{k+1,l} &= B_{kl}^{k-1,l} = \eta, \\ B_{kl}^{k,l+1} &= B_{kl}^{k,l-1} = \lambda, \\ B_{kl}^{k+1,l+1} &= B_{kl}^{k+1,l-1} = B_{kl}^{k-1,l+1} = B_{kl}^{k-1,l-1} = \mu, \\ B_{kl}^{kl} &= \xi. \end{aligned} \quad (2.7)$$

Expanding (2.3) by use of (2.5) we get from (2.7)

$$\begin{aligned} \eta &= x_3 y_2 = x_1 y_2, \\ \lambda &= y_1 x_2 = y_3 x_2, \\ \mu &= x_1 y_1 = x_3 y_1 = x_1 y_3 = x_3 y_3, \\ \xi &= x_2 y_2. \end{aligned} \quad (2.8)$$

These nine equations in six unknowns imply the restrictions

$$\eta \lambda = \mu \xi, \quad (2.9)$$

$$x_1 = x_3, \quad y_1 = y_3. \quad (2.10)$$

It should be noted that if μ is zero, as for a five-point scheme, then by (2.9) either λ or η must also be zero so that no factorable five-point scheme is possible.

More generally it can be shown that if we let

$$B_{kl}^{ij} = w_{kl}^{mn} b_{mn}^{ij}$$

with $w_{kl}^{mn} = 0$ when $k - m > 1$, $k - m < 0$, $l - n > 1$, or $l - n < 0$ and $b_{mn}^{ij} = 0$ when $i - m > 1$, $i - m < 0$, $j - n > 1$, or $j - n < 0$, then either (2.9) or $\xi = 4\mu$ must hold.

To obtain a proper representation of the Laplacian, we consider the Taylor series expansion of θ ,

$$\begin{aligned} \theta(x + \Delta x, y + \Delta y) = & \theta(x, y) + b\Delta x + c\Delta y + d(\Delta x)^2 + e(\Delta x)(\Delta y) + f(\Delta y)^2 \\ & + g(\Delta x)^3 + h(\Delta x)^2\Delta y + i(\Delta x)(\Delta y)^2 + j(\Delta y)^3. \end{aligned} \quad (2.11)$$

Applying the difference scheme (2.7) to (2.11) we obtain

$$B_{kl}^{ij} \theta_{i,j} = (4\mu + 2\eta + 2\lambda + \xi) \theta_{kl} + (4\mu + 2\eta) d(\Delta x)^2 + (4\mu + 2\lambda) f(\Delta y)^2. \quad (2.12)$$

As

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 2(d + f), \quad (2.13)$$

we must have, by (2.6),

$$(2\mu + \eta)(\Delta x)^2 = 1, \quad (2.14a)$$

$$(2\mu + \lambda)(\Delta y)^2 = 1, \quad (2.14b)$$

$$4\mu + 2\eta + 2\lambda + \xi = \beta. \quad (2.14c)$$

Solving (2.9) and (2.14) we obtain

$$\mu = 1/[\beta(\Delta x)^2(\Delta y)^2], \quad (2.15a)$$

$$\eta = (\Delta x)^{-2} \{1 - 2/[\beta(\Delta y)^2]\}, \quad (2.15b)$$

$$\lambda = (\Delta y)^{-2} \{1 - 2/[\beta(\Delta x)^2]\}, \quad (2.15c)$$

$$\xi = \beta \{1 - 2/[\beta(\Delta y)^2]\} \{1 - 2/[\beta(\Delta x)^2]\}. \quad (2.15d)$$

We may now solve (2.8) and (2.15) for x_i and y_i . Thus,

$$\begin{aligned} B_{kl}^{ij} = & [\beta(\Delta x)^2(\Delta y)^2]^{-1} \{ \delta_k^{i+1} + [\beta(\Delta x)^2 - 2] \delta_k^i + \delta_k^{i-1} \} \\ & \times \{ \delta_l^{j+1} + [\beta(\Delta y)^2 - 2] \delta_l^j + \delta_l^{j-1} \}. \end{aligned} \quad (2.16)$$

Or,

$$B_{kl}^{ij} = [\beta(\Delta x)^2(\Delta y)^2]^{-1} A_k^i B_l^j, \quad (2.17)$$

where, letting

$$\begin{aligned} u &= \beta(\Delta x)^2 - 2, \\ v &= \beta(\Delta y)^2 - 2, \end{aligned} \quad (2.18)$$

we define

$$A_k^i = \begin{bmatrix} u & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & u & 1 & 0 & 0 & & & \\ 0 & 1 & u & 1 & 0 & & & \\ 0 & 0 & 1 & u & 1 & & & \\ 0 & 0 & 0 & 1 & u & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ \cdot & & & & & & & \cdot \end{bmatrix} \quad (2.19)$$

and

$$B_i^i = \begin{bmatrix} v & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & v & 1 & 0 & 0 & & & \\ 0 & 1 & v & 1 & 0 & & & \\ 0 & 0 & 1 & v & 1 & & & \\ 0 & 0 & 0 & 1 & v & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ \cdot & & & & & & & \cdot \end{bmatrix} \quad (2.20)$$

We may factor A_k^i and B_i^i as follows:

$$A_k^i = {}_x w_k^m {}_x b_m^i = \begin{bmatrix} u - s_0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & u - s_1 & 0 & 0 & 0 & & & \\ 0 & 1 & u - s_2 & 0 & 0 & & & \\ 0 & 0 & 1 & u - s_3 & 0 & & & \\ 0 & 0 & 0 & 1 & u - s_4 & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ \cdot & & & & & & & \cdot \end{bmatrix} \quad (2.21)$$

$$\times \begin{bmatrix} 1 & s_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & s_2 & 0 & 0 & & & \\ 0 & 0 & 1 & s_3 & 0 & & & \\ 0 & 0 & 0 & 1 & s_4 & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ \cdot & & & & & & & \cdot \end{bmatrix},$$

where

$$s_0 = 0 \quad s_k = (u - s_{k-1})^{-1} \quad (2.22)$$

and

$$B_i^j = {}_v w_l^m {}_v b_m^j = \begin{bmatrix} v - r_0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & v - r_1 & 0 & 0 & 0 & & & \\ 0 & 1 & v - r_2 & 0 & 0 & & & \\ 0 & 0 & 1 & v - r_3 & 0 & & & \\ 0 & 0 & 0 & 1 & v - r_4 & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ \cdot & & & & & & & \cdot \end{bmatrix} \quad (2.23)$$

$$\times \begin{bmatrix} 1 & r_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & r_2 & 0 & 0 & & & \\ 0 & 0 & 1 & r_3 & 0 & & & \\ \cdot & & & \cdot & & & & \\ 0 & 0 & 0 & 1 & r_4 & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ \cdot & & & & & & & \cdot \end{bmatrix},$$

where

$$r_0 = 0 \quad r_l = (v - r_{l-1})^{-1}. \quad (2.24)$$

If we now define

$${}^n g_{v\omega} = {}_v b_v^i {}_v b_\omega^j {}^n \theta_{ij} \quad (2.25)$$

and

$$w_{kl}^{v\omega} = {}_v w_k^v {}_v w_l^\omega \quad (2.26)$$

then (2.2) becomes, by (2.17), (2.21), and (2.23),

$$w_{kl}^{v\omega} {}^n g_{v\omega} = \beta(\Delta x)^2 (\Delta y)^2 {}^n a_{kl}. \quad (2.27)$$

Now as $w_{kl}^{v\omega}$ involves only values of $(v\omega)$ such that $v \leq k$, $\omega \leq l$, we may proceed from low index numbers to high ones and solve (2.27) for the ${}^n g_{v\omega}$ by straightforward elimination. Then as ${}_v b_v^i {}_v b_\omega^j$ involves only values of (ij) such that $i \geq v$, $j \geq \omega$, we may proceed from high index numbers to low ones and solve (2.25) for the ${}^n \theta_{ij}$ by straightforward elimination. The formulae involved may be written conveniently as

$${}^n g_{ii} = \frac{\beta(\Delta x)^2 (\Delta y)^2 {}^n a_{ii} - {}^n g_{i-1,i-1} - (v - r_{i-1}) {}^n g_{i-1,i} - (u - s_{i-1}) {}^n g_{i,i-1}}{(v - r_{i-1})(u - s_{i-1})}, \quad (2.28)$$

$${}^n\theta_{ij} = {}^ng_{ij} - s_i {}^n\theta_{i+1,j} - r_j {}^n\theta_{i,j+1} - s_i r_j {}^n\theta_{i+1,j+1} \quad (2.29)$$

with appropriate modifications at the boundaries.

3. Generalization to the non-linear case. In this section we shall consider generalizing the method given in Sec. 2 to the solution of non-linear partial differential equations of the type

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = h(x, y, \psi) \frac{\partial \psi}{\partial t}. \quad (3.1)$$

In the linear case we modified the equation under consideration by adding $\beta\psi$ to each side. Here we obtain

$$\beta\psi + \nabla^2 \psi = \beta\psi + h(x, y, \psi) \frac{\partial \psi}{\partial t}. \quad (3.2)$$

In the linear case we chose β so that the right-hand side was independent of θ at the advanced time. Here this is not possible. Let us, however, choose

$$0 > \beta \geq -\lambda \text{ Minimum}_{i,j} \left\{ h(x_i, y_j, \psi_{ij}) + \psi_{ij} \frac{\partial h}{\partial \psi} \bigg|_{ij} \right\}, \quad (3.3)$$

where λ is the coefficient of ψ at the advanced time in the representation of the time derivative. In the heat flow problem h will be positive and in most cases of interest (see Sec. 6) will vary roughly like $\psi^{-\gamma}$, where $0 \leq \gamma < 1$. Let us now guess a value of ψ , and substitute it in the right side of the difference equation approximation to (3.2). We may now calculate by the method of Sec. 2 the values of ψ_{ij} which appear on the left side. If our guess was close to the true solution, we may expand everything to first order in the error. Let

$$\psi_{\text{guess}} = \psi_{\text{true}} + \epsilon^* \quad (3.4)$$

$$\psi_{\text{calculated}} = \psi_{\text{true}} + \epsilon.$$

Then, by (3.2)

$$\beta\epsilon_{ij} + \nabla^2 \epsilon_{ij} = \left[\beta + \lambda h(ij) + \lambda \Delta_{ij} \frac{\partial h}{\partial \psi} \bigg|_{ij} \right] \epsilon_{ij}^*, \quad (3.5)$$

where $\lambda \Delta_{ij}$ is the representation of the time derivative. Let us restrict the possible choice of representations of the time derivative such that (all $\psi \geq 0$)

$$({}^n\Delta_{ij})/({}^n\psi_{ij}) \leq 1. \quad (3.6)$$

This condition will be satisfied if, for instance,

$${}^n\Delta_{ij} = {}^n\psi_{ij} - {}^{n-1}\psi_{ij} \quad (3.7)$$

and it is satisfied for our choice (4.1) as long as the temperature does not drop by a factor of more than 4 at one time step. We shall neglect $\nabla^2 \epsilon_{ij}$ as small in (3.5) because we expect our guess to have only small systematic errors, rather than the type of errors which would create large errors in the second derivative. Thus,

$$\epsilon_{ij} \approx \left\{ 1 + \left[\lambda h(ij) + {}^n\Delta_{ij} \frac{\partial h}{\partial \psi} \bigg|_{ij} \right] / \beta \right\} \epsilon_{ij}^* = -\chi_{ij} \epsilon_{ij}^*. \quad (3.8)$$

Because of our selection (3.3) of β , χ must be positive or zero. Therefore, if we take a weighted average of ψ_{guess} and $\psi_{\text{calculated}}$ as

$$\psi' = \frac{\psi_{\text{calculated}} + \chi \psi_{\text{guess}}}{1 + \chi}, \quad (3.9)$$

then ψ' will agree with ψ_{true} to within second order in ϵ^* . If ψ' is now used as a new guess, we may continue our iteration procedure and be assured of second-order convergence. We shall see in Sec. 6 that this scheme provides very rapid convergence in a sample case.

4. Stability. Let us first consider the stability of our method as described in Sec. 2 for the case of linear flow. We pick a representation of $\partial\theta/\partial t$ which, to within terms of third order, gives $\partial\theta/\partial t$ evaluated at the advanced time, namely,

$$(1.5\theta^n - 2\theta^{n-1} + 0.5\theta^{n-2})/\Delta t. \quad (4.1)$$

From (2.1) and (2.6) we see that β of Sec. 2 is

$$\beta = -3/(2\alpha^2 \Delta t). \quad (4.2)$$

Let us obtain the exact solutions to our difference equations. We may expand any function on the mesh points in a Fourier series, so it will be sufficient to consider the behavior of

$$\theta_{ij} = \theta_0 \exp(ib_1 x_i + ib_2 y_j) X(n). \quad (4.3)$$

We shall assume that

$$X(n) = Z^{n/\Delta t}. \quad (4.4)$$

Substituting (4.3) into (2.1) as expressed by (2.16) we obtain, making use of trigonometric half-angle formulae,

$$\begin{aligned} \Gamma &= [-\beta(\Delta x)^2 + 4 \sin^2(\frac{1}{2}b_1 \Delta x)][-\beta(\Delta x)^2 + 4 \sin^2(\frac{1}{2}b_2 \Delta y)](\beta \Delta x \Delta y)^{-2} \\ &= (4Z^{-1} - Z^{-2})/3. \end{aligned} \quad (4.5)$$

Thus,

$$Z_{\pm}^{-1} = 2 \pm (4 - 3\Gamma)^{1/2}. \quad (4.6)$$

By definition, as b_1 and b_2 are real, $\Gamma > 1$. Thus, for $\Gamma \leq 4/3$, Z_{\pm} are both real and

$$1 > Z_- \geq \frac{1}{2} \geq Z_+ > \frac{1}{3}. \quad (4.7)$$

If $\Gamma > 4/3$, then Z_{\pm} are both complex and

$$|Z_{\pm}| = (3\Gamma)^{-1/2} < \frac{1}{2}. \quad (4.8)$$

Therefore as $|Z_{\pm}| < 1$, the difference scheme is unconditionally stable.

In the limit as Δx , Δy , and Δt tend to zero,

$$\begin{aligned} (Z_-)^{1/\Delta t} &\rightarrow \exp[-\alpha^2(b_1^2 + b_2^2)t], \\ (Z_+)^{1/\Delta t} &\rightarrow 3^{-t/\Delta t} \exp[\frac{1}{3}\alpha^2(b_1^2 + b_2^2)t]. \end{aligned} \quad (4.9)$$

The root Z_- represents the analytic solution of (2.1). The root Z_+ enables us to represent any computational error involved in advancing from time $n - 2$ to $n - 1$. We see from (4.7) and (4.8) that this error is damped by at least a factor of 2 at each time step.

The behavior of the solution of the difference equations in the limit as $t \rightarrow \infty$ with $\Delta t \rightarrow 0$ is also of interest. This behavior may be obtained by setting $Z = 1$ in (4.5) and removing the reality requirement on b_1 and b_2 . Thus, for the steady state

$$(\Delta x)^2 [4 \sin^2 (\frac{1}{2} b_1 \Delta x)]^{-1} + (\Delta y)^2 [4 \sin^2 (\frac{1}{2} b_2 \Delta y)]^{-1} = -2\alpha^2 \Delta t / 3. \quad (4.10)$$

If we have enough mesh points so that $\Delta x b_1 \ll 1$, $\Delta y b_2 \ll 1$, then we may expand the sines to first order and by some manipulation obtain

$$(b_1^2 + b_2^2) / b_1^2 \approx 2b_1^2 \alpha^2 \Delta t / (3 + 2b_1 \alpha^2 \Delta t). \quad (4.11)$$

In the differential equation, the steady state solution is characterized by

$$b_1^2 + b_2^2 = 0. \quad (4.12)$$

In order to get a good steady solution we see from (4.11) and (4.12) that $b_1^2 \alpha^2 \Delta t$ must be small. In order to get a good time-dependent solution we must be near the limit given in (4.9). This requires that $b_1 \Delta x$ and $b_2 \Delta y$ be small, and $b_1^2 \alpha^2 \Delta t$ be small. Hence, we will get a good asymptotic solution when the requirements are met for a good time-dependent solution. We see from the above analysis that this method is limited in mesh size and time-step length only by the requirements of accuracy and not by stability.

In the non-linear case, the analysis is more difficult but proceeds in a similar manner. We shall not investigate stability in the general case (3.1) but shall restrict h to be of the form

$$h(x, y, \psi) = f(x, y) \psi^\gamma, \quad (4.13)$$

where $0 \geq \gamma > -2$. We shall for convenience consider only the case of homogeneous boundary conditions. We can, of course, simulate non-homogeneous boundary conditions by letting f be very large near the boundary and so make a region near the boundary an effective heat reservoir. In this analysis we follow Bellman [3] and extend his analysis to the non-linear case. We remark that it is both necessary and sufficient for stability in the usual sense that

$$\lim_{n \rightarrow \infty} \sum_{i,j} h(i, j, {}^n \psi_{ij}) ({}^n \psi_{ij})^2 = 0 \quad (4.14)$$

hold for any initial conditions and homogeneous boundary conditions. We shall prove that our differencing scheme is unconditionally stable in the sense that (4.14) holds for any $[\Delta t / (\Delta x)^2]$.

Let us fix our attention on a time, which we shall denote by n . Let us attempt to separate variables. From (2.17) and (3.1) we obtain the equation which must be satisfied for some separable component, v . It is

$$\begin{aligned} \{ [\beta (\Delta x)^2 (\Delta y)^2]^{-1} A_k^i B_l^j - \beta \delta_k^i \delta_l^j \} {}^n \psi_{ij}(v) &= h(k, l, {}^n \psi_{kl}) {}^n \psi_{kl}(v) \\ &\times [3 - 4Z^{-1}(v) + Z^{-2}(v)] / (2\epsilon \Delta t), \end{aligned} \quad (4.15)$$

where we assume that

$${}^n \psi_{ij}(v) = Z(v) {}^{n-1} \psi_{ij}(v) \quad (4.16)$$

and that

$${}^n \psi_{ij} = \sum_v {}^n \psi_{ij}(v) p(v). \quad (4.17)$$

In fixing our attention on a time n we imagine that ${}^n\psi_{ij}$ is somehow known. If we set

$$[3 - 4Z^{-1}(v) + Z^{-2}(v)]/(2 \Delta t) = \lambda \quad (4.18)$$

then it is easy to show that the solution of (4.15) is the same as the solution of the following problem. Let (suppressing the n and v)

$$F \equiv \sum_{k,l} \left[\left(\frac{\psi_{k+1,l} - \psi_{kl}}{\Delta x} \right)^2 + \left(\frac{\psi_{k,l+1} - \psi_{kl}}{\Delta y} \right)^2 - \frac{(\psi_{k+1,l+1} - \psi_{k+1,l} - \psi_{k,l+1} + \psi_{kl})^2}{\beta(\Delta x)^2(\Delta y)^2} \right] \quad (4.19)$$

and

$$G \equiv \sum_{k,l} h(k, l, \psi_{kl})(\psi_{kl})^2 = 1. \quad (4.20)$$

Find the extrema of $F + \lambda G$ subject to (4.20) and homogeneous boundary conditions. If we have $N \times M$ interior mesh points, then the theory of quadratic forms [4] assures us, as β is negative, that there exist NM orthogonal vectors $[{}^n\psi_{kl}(v)]$ which satisfy (4.19) and (4.20) and therefore (4.15). The orthogonality is in the sense that

$$({}^n\psi_{kl}(v), {}^n\psi_{kl}(\mu)) \equiv \sum_{k,l} h(k, l, \psi_{kl}) {}^n\psi_{kl}(v) {}^n\psi_{kl}(\mu) = \delta_{v\mu}. \quad (4.21)$$

As λ can be shown to be the negative of F , we must have

$$\lambda < 0. \quad (4.22)$$

We may now solve (4.18) for the corresponding values of $Z_*(v)$. Equation (4.18) implies that for the Γ of (4.5) we get

$$\Gamma = 1 - 2\lambda \Delta t/3 > 1 \quad (4.23)$$

by (4.22). Thus $|Z_*| < 1$ as in the linear case. Let us now expand ${}^n\psi_{ij}$ in terms of our set of orthogonal vectors $[{}^n\psi_{ij}(v)]$. In order to reproduce ${}^{n-1}\psi_{ij}$ and ${}^{n-2}\psi_{ij}$ it will be necessary to use parts proportional to Z_+ and to Z_- ; however, we have enough freedom to effect this decomposition as ${}^n\psi_{ij}$ is assumed to be calculated by our difference scheme from the values at the two previous times. Let us now compute the time derivative of the Norm of ${}^n\psi_{ij}$

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \sum_{k,l} h(k, l, {}^n\psi_{kl}) ({}^n\psi_{kl})^2 \right\} &= \sum_{k,l} f(k, l) \frac{\partial}{\partial t} ({}^n\psi_{kl})^{2+\gamma} \\ &= (2 + \gamma) \sum_{k,l} h(k, l, {}^n\psi_{kl}) {}^n\psi_{kl} \frac{\partial {}^n\psi_{kl}}{\partial t}. \end{aligned} \quad (4.24)$$

If we now approximate $\partial\psi/\partial t$ by an expression of the form of (4.1) and use our expansion in terms of ${}^n\psi_{ij}(v)$, then, if $p(v)$ denotes the coefficient of ${}^n\psi_{ij}(v)$ in ${}^n\psi_{ij}$, we get

$$\begin{aligned} \frac{\partial}{\partial t} \text{Norm } ({}^n\psi_{ij}) &= (2 + \gamma) \sum_v |p(v)|^2 \lambda(v) \\ &\leq \text{Max}_v [\lambda(v)] (2 + \gamma) \text{Norm } ({}^n\psi_{ij}). \end{aligned} \quad (4.25)$$

We must now examine the behavior of $\text{Max}_v [\lambda(v)]$. First let us consider the maximum attained by $\text{Max}_v [\lambda(v)]$ with respect to all variations of components for a fixed Norm.

As this represents continuous variation over a closed and bounded set, the maximum of $\text{Max}_v [\lambda(v)]$ must be attained at some point of the set by Weierstrass' theorem. However, applying the arguments that lead to (4.22) we see that for fixed Norm,

$$\lambda(v) \leq \lambda(\text{Norm}) < 0. \quad (4.26)$$

We may consider a variation in Norm by simply scaling all the ψ_{kl} . It easily follows from (4.19) and (4.20) that for any Norm

$$\lambda(v) \leq \lambda_0 [\text{Norm} (" \psi_{ij})]^{-\gamma}. \quad (4.27)$$

Thence, by (4.25) and (4.27) we have

$$\frac{\partial}{\partial t} [\text{Norm} (" \psi_{ij})] \leq \lambda_0 (2 + \gamma) [\text{Norm} (" \psi_{ij})]^{1-\gamma}. \quad (4.28)$$

Thus as $0 \geq \gamma > -2$, and $\lambda_0 < 0$, we see that the Norm must decrease to zero. In the linear case ($\gamma = 0$) this decrease is exponential. This proof is subject only to the proviso that Δt be small enough so that (4.1) will give us an approximation of the time derivative of the Norm that reproduces the sign properly. As the norm tends to zero, we see that our scheme is unconditionally stable, at least for this special class of problems. It should be noted that both the diffusion equation [2] and the radiation transport equation with power-law, temperature-dependent opacity belong to this class.

Analogous arguments for the non-linear case corresponding to the arguments involving (4.10) to (4.12) indicate that the requirements for obtaining good time-dependent and steady-state solutions are about the same as the requirements for obtaining such solutions in the linear case. Therefore, the non-linear case has the same properties so far as stability and accuracy are concerned as the linear case. The only way in which the numerical solution to the non-linear case differs essentially from the linear case in rectangular coordinates is that an iterative procedure must be used on each time-step in the non-linear case.

5. Extension to non-rectangular regions and three-dimensional problems with an axis of symmetry. Consider a simply-connected, two-dimensional region with more than one boundary point. Riemann's theorem [5] implies that we may map it by a one-to-one conformal transformation into a rectangular region. This transformation amounts to a change of independent variables. If

$$u = u(x, y) \quad v = v(x, y) \quad (5.1)$$

is such a conformal mapping, then (3.1) is transformed [6] into

$$g^2 \left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} \right) = h(u, v, \psi) \frac{\partial \psi}{\partial t}, \quad (5.2)$$

where

$$\begin{aligned} g^2 &= \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \\ &= \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \\ &= g^2(u, v). \end{aligned} \quad (5.3)$$

Thus, defining

$$H(u, v, \psi) = h(u, v, \psi)/g^2(u, v). \quad (5.4)$$

we have

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= h(x, y, \psi) \frac{\partial \psi}{\partial t} \\ \rightarrow \frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} &= H(u, v, \psi) \frac{\partial \psi}{\partial t}. \end{aligned} \quad (5.5)$$

We may therefore first transform our region into a rectangle and then solve by the method of Sec. 3. It should be noted that when a non-rectangular region is to be solved, the iterative scheme of Sec. 3 is necessary, even in the linear case.

One simple example of this method is a transformation to polar coordinates. Let

$$u + iv = \log(x + iy) \quad (5.6a)$$

or

$$u = \log r \quad v = \varphi, \quad (5.6b)$$

where r and φ are the standard polar coordinate radius and angle. Equation (2.1) becomes

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = e^{2u} \alpha^{-2} \frac{\partial \theta}{\partial t}. \quad (5.7)$$

The singularity near the origin ($u \rightarrow -\infty$ as $r \rightarrow 0$) should be noted. This transformation actually carries a slit annulus into a rectangle. There are other more complicated transformations which will carry a whole circle into a rectangle. For instance,

$$u + iv = \int_0^{x+iy} (1-w^4)^{-1/2} dw. \quad (5.8)$$

For problems with an axis of symmetry, we adopt cylindrical coordinates as the basic point of departure. The Laplacian is

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2}. \quad (5.9)$$

As we assume an axis of symmetry we may choose our coordinate system so ψ is independent of φ . Let us set

$$\omega = r^{1/2} \psi. \quad (5.10)$$

Thus Eq. (3.1) becomes

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} = h(r, z, \omega) \frac{\partial \omega}{\partial t} - \frac{\omega}{4r^2}. \quad (5.11)$$

From the point of view of numerical solution we are free to add a term involving only ω and independent of any derivatives, and the method of solution proposed in Sec. 3 is still applicable. It should be noted that the iterative scheme will have to be modified to include this term, and if the coefficient of ω at the advanced time contributed by the right-hand side of the difference equation analogue of (5.11) may change sign, we may

no longer pick β so that the error will change sign everywhere between the guessed and calculated values of α , but we must in some regions extrapolate for a new guess of ω , rather than interpolate throughout as we did in Sec. 3.

The boundary condition to be applied along the axis of symmetry is that

$$\frac{d(\log \omega)}{d(\log r)} = \frac{1}{2}. \quad (5.12)$$

This condition ensures that ψ will be finite on the axis.

For many problems with an axis of symmetry, cylindrical coordinates are not the most convenient. Other axial shapes may be treated by conformally mapping them into a rectangle. A sphere is conveniently treated by transforming (5.11) by (5.6) where we, of course, consider only the right half $r - x$ plane ($-\pi/2 \leq v \leq \pi/2$). Thus for spherical coordinates we get

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} = e^{2u} h(u, v, \omega) \frac{\partial \omega}{\partial t} - \frac{\omega}{4}. \quad (5.13)$$

6. Numerical examples. In order to illustrate the method described above we programmed it for an IBM 704. Two sample calculations are described.

We set up initial conditions corresponding to the solution of the linear problem,

$$\theta(x, y, t) = \sin(\pi x/l) \sin(\pi y/l) \exp(-2\alpha^2 \pi^2 t/l^2). \quad (6.1)$$

We chose $\alpha^2 \Delta t / (\Delta x)^2 = 1.02392228$ so that the amplitude would be diminished by a factor of $10^{-1/16}$ at each time step. This choice enables us to check the accuracy at many times without excessive labor in calculating the analytic values. We ran this calculation with mesh points 15° apart (11×11 interior points), and advanced it 320 time steps. It was found that over the range of 20 decades through which the solution passed, the only discernible numerical error was a truncation error (the 704 does not round, but truncates instead) that caused the solution to decrease by an additional factor of $(1 - 5.44 \times 10^{-8})$ at each time step. When a fixed number of decimal places are carried, the truncation error is expected to be proportional to $\alpha^4 (\Delta t)^2 / [(\Delta x)^2 (\Delta y)^2]$. The difference between the solution of the difference equation and the differential equation is that time flows 2.5 per cent more rapidly in the latter case. It should be noted that the explicit scheme is unstable for this case. The calculating time for this case was about 6 minutes, or about 10 milliseconds per cycle point.

For our other example we chose radiation flow in a material medium. The equation is

$$\nabla^2(\theta^4) = 16K \frac{\partial \theta}{\partial t}. \quad (6.2)$$

If we let $\psi = K^{-4/3} \theta^4$, (6.2) becomes

$$\nabla^2 \psi = 4\psi^{-3/4} \frac{\partial \psi}{\partial t}. \quad (6.3)$$

[If there were a power-law, temperature-dependent opacity, the equation would be

$$\nabla \cdot (\theta^i \nabla \theta^4) = 16K \frac{\partial \theta}{\partial t}. \quad (6.4)$$

By use of the identity

$$\theta^m \nabla \theta^n = \frac{n}{n+m} \nabla \theta^{m+n} \quad (6.5)$$

and the transformation

$$\psi = \theta^{4+i} \quad (6.6)$$

we can put (6.4) in the form of (3.1).]

For our sample calculation we picked $\beta(\Delta x)^2 = \beta(\Delta y)^2 = -1.5$ and used 11×11 interior mesh points. The other constants were chosen so that this is an optimal iteration, as defined by (3.3). For initial conditions we chose

$$\psi(x, y) = 1 - \sin(2\pi x/l) \sin(2\pi y/l) \quad (6.7)$$

and we maintained the boundaries at unit temperature. We added 0.01 to ψ at the central point to prevent it from vanishing. Initially some 10 or 12 iterations were required per time step, but after 15 time steps, only about 2 iterations per time step were required for three-place accuracy. We make the initial guess of ψ at each time step by a linear extrapolation from the two previous times. The running time for this calculation was about 15 minutes for 120 time steps, or about 20 milliseconds per cycle point. From runs with other values of $\beta(\Delta x)^2$ we estimate that the accuracy of this solution is about 2 to 3 per cent.

It should be noted that some care must be exercised to prevent the guessing of a negative temperature, as the occurrence of temperatures of different signs produces an instability which causes the solution to diverge.

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BOOK REVIEWS

(Continued from p. 360)

Mathematik und Mechanik which is engaged, with the Association for Computing Machinery in the United States, in constructing an international algebraic language, to serve as a digital computer source language for mathematical problems. An agreement on terminology is a prerequisite for such an undertaking, and this work provides an acceptable basis, for translating programming terms between the English, French, Swedish, Dutch, and Russian languages. A cursory check revealed only a few omissions, e.g. object language, source language, mask, table look-up, assembly program.

WALTER FREIBERGER

Computational methods of linear algebra. By V. N. Faddeeva. Dover Publications, Inc., New York, 1959. x + 252 pp. \$1.95.

This textbook on numerical methods consists of three chapters:

- I. Basic material from linear algebra (62 pages).
- II. Systems of linear equations (84 pages).
- III. The proper numbers and proper vectors of a matrix (96 pages).

It is a translation of a noted Russian work and is unique in its clarity, elegance and scope. There is a wealth of useful numerical examples which, with the numerical tables, have been checked by the translator. The emphasis is on methods useful with desk calculators.

The first chapter presents the mathematical background, assuming knowledge of elementary determinant theory but not of matrices. Most proofs are shown and the author works in the complex field. The Jordan canonical form is given without proof, as is the theory of elementary divisors. There is a section on vector and matrix norms and on limits.

The second chapter starts with Gauss' elimination method in its various forms, recommends the square root method for symmetric systems and demonstrates partitioning and escalator methods. A welcome feature is a careful discussion of convergence criteria for the ordinary and the single-step (Seidel) iteration methods, with numerical illustrations, but gradient methods are omitted.

The third chapter treats, firstly, of various ways of determining the characteristic polynomial, methods associated with the names of A. N. Krylov, Samuelson, Danilevskii, Leverrier, Faddeev, and of the escalator and interpolation methods for obtaining the eigenvalues. The classical iteration method for finding the largest eigenvalue, with and without accelerated convergence, is described, with λ -differencing chosen as the method for determining the following eigenvalues.

The author has unfortunately not included a discussion of how automatic computing equipment affects the choice of techniques. If she had, she would no doubt have presented Jacobi's method for eigenvalue analysis. There is a bibliography of 40 titles and a useful index. The translation is readable and the book is invaluable for everyone with any occasion to analyse linear systems.

WALTER FREIBERGER

Operations research—methods and problems. By M. Sasieni, A. Yaspan, and L. Friedman. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1959. xi + 316 pp. \$10.25.

The purpose of this book is largely to provide a collection of illustrative problems for use in courses in operations research. The problems (many of which are in the form of worked examples) are primarily applications of elementary probability theory; they are classified by background topic under chapter headings of sampling, inventory, replacement, waiting lines, competitive strategies, allocation, and sequencing. A short general discussion is given at the beginning of each chapter.

The keynote of the book is simplicity, and it is refreshing to note the author's statement that "Although a great deal of elegant mathematics appears in the literature, far more problems have been

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AIRFOIL IN A SONIC SHEAR FLOW JET: A MIXED BOUNDARY VALUE PROBLEM FOR THE GENERALIZED TRICOMI EQUATION*

BY

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Abstract. In this paper, small perturbations of a non-uniform two-dimensional flow of a compressible inviscid fluid are considered. It is shown that for a particular class of main stream Mach number distributions, which are characterized by a sonic line along the x -axis, the linearized shear flow equation may be transformed into the generalized Tricomi equation. The mixed boundary value problem which results from considering perturbations generated by a two-dimensional camber surface is formulated and solved by utilizing the Wiener-Hopf technique.

1. Introduction. In recent years, Lighthill [1] Chang and Werner [2] and others have treated two-dimensional linearized compressible shear flow problems. In this paper a parallel but non-uniform stream of compressible fluid is perturbed by a thin two-dimensional airfoil which is located along a sonic line. The main stream Mach number distribution, $M(y)$, resembling that approached by a sharp edged sonic jet under the influence of viscosity, is assumed symmetrical about the sonic line at $y = 0$.

Let $p(x, y)$ be the perturbation pressure, P be the constant freestream pressure, and $\alpha(x, y)$ be the slope of the streamlines in the disturbed flow. Also let $M_0 = M(0)$. α and p , if small enough, can be determined from a function $\phi(x, y)$ satisfying the linear equation [3]

$$(1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{2}{M} \frac{dM}{dy} \frac{\partial \phi}{\partial y} = 0. \quad (1.1)$$

ϕ is related to $p(x, y)$ and $\alpha(x, y)$ by

$$\frac{\partial \phi}{\partial x} = \frac{p}{(\gamma/2) P M_0^2} = C_p, \quad (1.2)$$

$$\frac{\partial \phi}{\partial y} = -2 \frac{M^2}{M_0^2} \alpha. \quad (1.3)$$

Here C_p is the local pressure coefficient and γ the ratio of specific heats. It is generally accepted that (1.1) is a valid linearization of the equations of fluid motion provided that dM/dy is large enough at points where $M = 1$. This condition is not satisfied for the type of Mach number profile considered in this paper. It appears reasonable, however, that (1.1) is valid if higher derivatives of M are large enough at sonic points.

Let the airfoil have unit chord length and be a camber surface so that the slopes of the upper and lower surfaces both equal $\alpha_0(x)$. The condition that the flow be tangent to the airfoil can be obtained from (1.3) as

$$\frac{\partial \phi(x, 0)}{\partial y} = -2\alpha_0(x), \quad 0 < x < 1. \quad (1.4)$$

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For subsonic flows, the disturbance must die out at infinity. This condition can be accomplished by imposing that ϕ approaches zero at infinity. The problem can be somewhat simplified by specifying boundary data along the whole x -axis instead of only on the airfoil. It is shown in Appendix I that, as in incompressible potential flow ϕ has a jump discontinuity which is equal to the lift coefficient, C_l . This jump can be taken conveniently to be along the positive x -axis behind the airfoil. In addition, because of the symmetry of the Mach number profile and the anti-symmetry of the airfoil, the transformation $y \rightarrow -y$, $\phi \rightarrow -\phi$ leaves Eqs. (1.1) and (1.4) unchanged. This implies that ϕ is an odd function of y . This fact and the jump condition require

$$\begin{aligned}\phi(x, 0_+) &= 0 & x < 0 \\ &= -C_l/2 & x > 0.\end{aligned}\quad (1.5)$$

Equations (1.4) and (1.5) give mixed boundary data along the entire x -axis.

In general, solutions of (1.1) are difficult to obtain with any arbitrarily given $M(y)$. However, as will be shown in Part II, for a particular choice of $M(y)$, Eq. (1.1) can be transformed to the form

$$Y^n \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial Y^2} = 0, \quad Y \geq 0, \quad (1.6)$$

where n is a non-negative constant and Y is a new independent variable

$$Y = b \int_0^y M^2(y) dy \quad b = \text{constant}.$$

Eq. (1.6) is commonly known as the generalized Tricomi equation. This equation has been widely studied as shown in the bibliography to [4]. However, no solution is known for the above mixed boundary conditions.

The present treatment will be divided into two parts. In Part I, the mathematical solution to this problem will be derived. In Part II, the application to an airfoil in a sonic shear flow jet and the details of the corresponding Mach number profile will be considered.

PART I. SOLUTION OF THE GENERALIZED TRICOMI EQUATION WITH MIXED BOUNDARY CONDITIONS

2. Statement of problems and procedure for solution. The problem is to solve the generalized Tricomi equation

$$Y^n \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial Y^2} = 0, \quad Y \geq 0, \quad n \geq 0 \quad (2.1)$$

with the mixed boundary conditions

$$\begin{aligned}\phi(x, 0) &= 0, & x < 0; \\ \phi(x, 0) &= -\frac{C}{2}, & x > 0; \\ \frac{\partial \phi(x, 0)}{\partial Y} &= \theta(x), & 0 < x < 1; \\ \phi(x, \infty) &= 0.\end{aligned}\quad (2.2)$$

Here C is a constant and $\theta(x)$ is bounded and integrable. If $\phi(x, 0)$ were given along the whole x -axis, the problem would be a Dirichlet problem which could be solved easily by use of classical techniques such as the Fourier integral or two-sided Laplace transforms. By applying Wiener-Hopf techniques [5, 6] to the present problem, the missing Dirichlet data, $\phi(x, 0)$ on the interval $(0, 1)$, can be found. This is carried out in two steps. In Sec. 3 a related auxiliary problem is first solved, namely, the generalized Tricomi equation with $\partial\phi(x, 0)/\partial Y$ specified on the positive x -axis, and $\phi(x, 0) = 0$ on the negative x -axis¹. This is done with the aid of two-sided Laplace transforms and the Wiener-Hopf technique. In the main problem $\partial\phi(x, 0)/\partial Y$ is given on $(0, 1)$ and $\phi(x, 0)$ is specified on $(1, \infty)$, consequently, the auxiliary problem gives an integral equation for the unknown quantities $\phi(x, 0)$ on $(0, 1)$ and $\partial\phi(x, 0)/\partial Y$ on $(1, \infty)$. In Sec. 4 this integral equation is solved by using Mellin transforms with the Wiener-Hopf technique.

3. Solution for $\phi(x, 0)$ on $(0, \infty)$ when $\partial\phi(x, 0)/\partial Y$ is specified on $(0, \infty)$ and $\phi(x, 0) = 0$ on $(-\infty, 0)$. If $\phi(x, Y)$ is a function which satisfies

$$Y^n \frac{\partial^1 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial Y^2} = 0, \quad Y \geq 0, \quad n \geq 0 \quad (3.1)$$

and the boundary conditions

$$\begin{aligned} \phi(x, 0) &= 0, & x < 0; \\ \frac{\partial\phi(x, 0)}{\partial Y} &= \theta_1(x), & x > 0; \\ \phi(x, \infty) &= 0, \end{aligned} \quad (3.2)$$

it will be shown that on the positive x -axis, the solution is

$$\phi(x, 0) = \frac{a^{-1}}{\Gamma^2(1/n + 2)} \int_0^x (x - \xi)^{(1/n+2)-1} d\xi \int_\xi^\infty \theta_1(\eta)(\eta - \xi)^{(1/n+2)-1} d\eta, \quad (3.3)$$

where

$$a = -(1/n + 2)^{+n/n+2} \Gamma(n + 1/n + 2) / \Gamma(1/n + 2).$$

First, let the boundary conditions (3.2) be restated as

$$\begin{aligned} \phi(x, 0) &= f_1(x)H(x), \\ \frac{\partial\phi(x, 0)}{\partial Y} &= \theta_1(x)H(x) + \theta_2(x)H(-x), \\ \phi(x, \infty) &= 0, \end{aligned} \quad (3.4)$$

where $f_1(x)$ and $\theta_2(x)$ are unknown functions and $H(x)$ is the Heaviside step function, i.e.,

$$\begin{aligned} H(x) &= 1, & x > 0 \\ &= 0, & x < 0. \end{aligned}$$

¹This problem differs from that of Carrier [7] in that Carrier's boundary data are given on a line perpendicular to the parabolic (sonic) line.

Secondly, let

$$\Phi(s, Y) = \int_{-\infty}^{\infty} e^{-sx} \phi(x, Y) dx \quad (3.5)$$

be the two-sided Laplace transform of $\phi(x, Y)$. Taking the Laplace transform of (1.1) and (3.4) gives

$$\begin{aligned} Y^n s^2 \Phi + \frac{\partial^2 \Phi}{\partial Y^2} &= 0 \\ \frac{\partial \Phi(s, 0)}{\partial Y} &= \Theta_1(s)_+ + \Theta_2(s)_-, \\ \Phi(s, 0) &= F_1(s)_+, \\ \Phi(s, \infty) &= 0, \end{aligned} \quad (3.6)$$

where $\Theta_1(s)_+$, $\Theta_2(s)_-$ and $F_1(s)_+$ are defined by

$$\Theta_1(s)_+ = \int_0^{\infty} e^{-sx} \theta_1(x) dx,$$

$$\Theta_2(s)_- = \int_{-\infty}^0 e^{-sx} \theta_2(x) dx,$$

$$F_1(s)_+ = \int_0^{\infty} e^{-sx} f_1(x) dx.$$

Some imposed properties of these functions and their consequences are given below:

(i) The given function $\theta_1(x)$ is assumed to approach zero like $\exp(-\epsilon x)$ as $x \rightarrow \infty$; thus $\Theta_1(s)_+$ will be analytic for $\operatorname{Re}(s) > -\epsilon$ where $\epsilon > 0$. Since $\theta_1(x)$ is a bounded function, $\Theta_1(s)_+ = O(|s|^{-1})$ as $s \rightarrow \infty$.

(ii) $\Theta_2(s)_-$, an unknown function which is to be determined, must approach zero algebraically as $x \rightarrow -\infty$ so that $\theta_2(s)_-$ will exist for $\operatorname{Re}(s) = 0$ and be analytic for $\operatorname{Re}(s) < 0$. The assumption $\theta_2(x) = O(-x)^{-p}$, $p < 1$ as $x \rightarrow 0$ assures that $\theta_2(x)$ is integrable near the origin. This condition implies that $\Theta_2(s)_- = O(|s|^{-1+p})$ as $|s| \rightarrow \infty$.

(iii) $F_1(s)_+$ is also an unknown function to be evaluated. The assumption here is that $f_1(x) = O(\exp(-\epsilon x))$ as $x \rightarrow \infty$ and is integrable near the origin. These conditions imply that $F_1(s)_+$ is analytic for $\operatorname{Re}(s) > -\epsilon$ and $O(|s|^{-1+q})$, $q > 0$ as $|s| \rightarrow \infty$.

The relevant solution of Eq. (3.6) is

$$\Phi = A Y^{1/2} K_{1/n+2} \left[(-s^2)^{1/2} \frac{2}{n+2} Y^{n+2/2} \right], \quad (3.7)$$

where s must be purely imaginary to satisfy the condition $\Phi \rightarrow 0$ as $Y \rightarrow \infty$. Differentiation of (3.7) yields

$$\frac{d\Phi}{dY} = -A (-s^2)^{1/2} Y^{n+1/2} K_{n+1/n+2} \left[(-s^2)^{1/2} \frac{2}{n+2} Y^{n+2/2} \right]. \quad (3.8)$$

Evaluating (3.7) and (3.8) at $Y = 0$ and using the boundary conditions from (3.6) we can write

$$\begin{aligned}\Phi(s, 0) &= \pi/2 \frac{(1/n + 2)^{-1/n+2} (-s)^{-1/2n+4}}{\sin \pi(n + 1/n + 2)\Gamma(1/n + 2)} A = F_1(s)_+, \\ \frac{d\Phi(s, 0)}{dY} &= -\pi/2 \frac{(1/n + 2)^{-(n+1/n+2)} (-s)^{-1/2n+4}}{\sin \pi(n + 1/n + 2)\Gamma(1/n + 2)} A = \Theta_1(s)_+ + \Theta_2(s)_-.\end{aligned}\quad (3.9)$$

Eliminating A from these two equations, we find,

$$aF_1(s)_+ = \frac{\Theta_1(s)_+ + \Theta_2(s)_-}{(-s^2)^{1/n+2}}, \quad (3.10)$$

where a is given in Eq. (3.3).

As it stands, Eq. (3.10) is valid only for purely imaginary s . In order to use the Wiener-Hopf technique, an expression which is analytic in a strip parallel to the imaginary axis is desired. Consider the function $(-s^2)^{1/n+2}$, $\text{Re}(s) = 0$ which is real and positive. How should this function be continued analytically into the complex s -plane? Consider the product $(-s)^{1/n+2}(s)^{1/n+2}$ for complex s . $(-s)^{1/n+2}$ is defined to be real when s is real and negative and a cut is introduced in the s plane along the positive real axis so that this function will be single-valued. $s^{1/n+2}$ is defined to be real when s is real and positive and a cut is introduced along the negative real axis. It is easily shown that when s is imaginary this product is equal to $(-s^2)^{1/n+2}$. The above product is taken as the continuation of $(-s^2)^{1/n+2}$. Now let a small separation of the branch points be made by writing $(-s)^{1/n+2}(s + \epsilon)^{1/n+2}$, $\epsilon > 0$. See Fig. 1. This is the same ϵ introduced in (i).

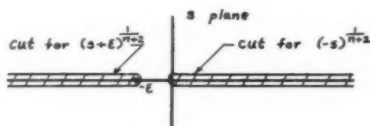


FIG. 1. Analytic region of $(-s)^{1/n+2}(s + \epsilon)^{1/n+2}$.

If this function is used in place of $(-s^2)^{1/n+2}$ in (3.10) we get

$$aF_1(s)_+(s + \epsilon)^{1/n+2} = \frac{\Theta_1(s)_+ + \Theta_2(s)_-}{(-s)^{1/n+2}}. \quad (3.11)$$

It is assumed that the solutions of this equation will satisfy (3.10) when ϵ goes to zero. This idea was used in [9]. Now the Wiener-Hopf technique can be used since both sides of Eq. (3.11) are analytic in the strip $-\epsilon < \text{Re}(s) < 0$.

We want to express $\Theta_1(s)_+ / (-s)^{1/n+2}$ as the sum of two functions, one analytic in the left half plane, the other analytic in the overlapping right half plane. By Cauchy's Integral Formula [11],

$$\begin{aligned}\frac{\Theta_1(s)_+}{(-s)^{1/n+2}} &= \frac{1}{2\pi i} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \frac{1}{\omega - s} \frac{\Theta_1(\omega)_+}{(-\omega)^{1/n+2}} d\omega \\ &\quad - \frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{1}{\omega - s} \frac{\Theta_1(\omega)_+}{(-\omega)^{1/n+2}} d\omega,\end{aligned}\quad (3.12)$$

where $-\epsilon < \gamma_1 < \text{Re}(s) < \gamma_2 < 0$. The first integral in Eq. (3.12) is analytic for $\text{Re}(s) < \gamma_2$, while the second is analytic for $\text{Re}(s) > \gamma_1$. Equation (3.12) can be

written as

$$\frac{\Theta_1(s)_+}{(-s)^{1/n+2}} = \sigma_2(s)_- - \sigma_1(s)_+, \quad (3.13)$$

where $\sigma_2(s)_-$, $\sigma_1(s)_+$ are defined by Eq. (3.12). Using Eq. (3.13), Eq. (3.11) becomes

$$af_1(s)_+(s + \epsilon)^{1/n+2} + \sigma_1(s)_+ = \frac{\Theta_2(s)_-}{(-s)^{1/n+2}} + \sigma_2(s)_- = E(s). \quad (3.14)$$

Since the two sides of this equation are analytic in overlapping half planes, they are sufficient to define an entire function $E(s)$. That is to say, the right hand side is the analytic continuation of the left hand side. From the estimates made in (i), (ii), (iii) and since γ_1 and γ_2 are of the order of $|s|^{-1}$, it is seen that

$$E(s)_+ = O(|s|^{(1/n+2)-1+q}),$$

$$E(s)_- = O(|s|^{(-1/n+2)-1+p}),$$

as $|s| \rightarrow \infty$, where $p, q < 1$. The derivative of E is bounded and goes to zero for large s . Therefore, by Liouville's theorem [11] E is a constant. But the above shows $E(s)_- \rightarrow 0$ as $|s| \rightarrow \infty$. This implies that $E = 0$. This fact allows the evaluation of $F_1(s)_+$. From Eq. (3.14),

$$F_1(s)_+ = -a^{-1} \frac{\sigma_1(s)_+}{(s + \epsilon)^{1/n+2}}, \quad \operatorname{Re}(s) > -\epsilon. \quad (3.15)$$

Let $\epsilon \rightarrow 0$ and substitute the integral expression for $\sigma_1(s)_+$. This gives

$$F_1(s) = a^{-1} s^{-1/n+2} \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \frac{1}{s - \omega} \frac{\Theta_1(\omega)_+}{(-\omega)^{1/n+2}} d\omega, \quad \operatorname{Re}(s) > 0, \quad (3.16)$$

where $\gamma_1 = 0$ since $\epsilon > \gamma_1 > 0$.

Taking the inverse transform of (3.16) we obtain

$$\phi(x, 0) = f_1(x) = a^{-1} L^{-1} \left\{ s^{-1/n+2} \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \frac{1}{s - \omega} \frac{\Theta_1(\omega)_+}{(-\omega)^{1/n+2}} d\omega \right\}. \quad (3.17)$$

The inverse transform² on the right hand side is the convolution of

$$L^{-1}(s^{-1/n+2}) = \frac{x^{(1/n+2)-1} H(x)}{\Gamma(1/n+2)}$$

with

$$L^{-1} \left\{ \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \frac{1}{s - \omega} \frac{\Theta_1(\omega)_+}{(-\omega)^{1/n+2}} d\omega \right\} = H(x) \frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{\omega x} \frac{\Theta_1(\omega)_+}{(-\omega)^{1/n+2}} d\omega \equiv H(x) \psi(x),$$

where $\psi(x)$ is defined by this equation. Use of the convolution integral (the inverse transform of the product of two transforms) gives

$$f_1(x) = a^{-1} \frac{1}{\Gamma(1/n+2)} \int_0^x (x - \xi)^{(1/n+2)-1} \psi(\xi) d\xi, \quad x > 0. \quad (3.18)$$

²All Laplace and Mellin transforms used in this paper can be found in [10].

But $\psi(\xi)$ is the convolution of

$$\frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{u\xi} \Theta_1(\omega)_+ d\omega = \theta_1(\xi) H(\xi)$$

with

$$\frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{u\xi} (-\omega)^{-1/n+2} d\omega = \frac{(-\xi)^{1/n+2} H(-x)}{\Gamma(1/n+2)}.$$

Therefore,

$$\psi(\xi) = \frac{1}{\Gamma(1/n+2)} \int_{\xi}^{\infty} \theta(\eta) (\eta - \xi)^{(1/n+2)-1} d\eta, \quad \xi > 0. \quad (3.19)$$

Substitution of Eq. (3.19) into Eq. (3.18) yields Eq. (3.3) as was to be shown.

4. The case where $\partial\phi(x, 0)/\partial y$ is given on $(0, 1)$. The result of Sec. 3 is

$$\phi(x, 0) = \frac{a^{-1}}{\Gamma^2(1/n+2)} \int_0^x (x - \xi)^{(1/n+2)-1} d\xi \int_{\xi}^{\infty} \theta_1(\eta) (\eta - \xi)^{(1/n+2)-1} d\eta, \quad (4.1)$$

$$0 < x < \infty,$$

where $\theta_1(x) = \partial\phi(x, 0)/\partial y$ is supposed to be known on $0 < x < \infty$. In the problem stated in Sec. 2, $\partial\phi(x, 0)/\partial y = \theta(x)$ is specified only for the interval $(0, 1)$ while $\phi(x, 0)$ is specified for $x > 1$. Hence Eq. (4.1) is an integral equation for the unknown functions. It will be shown that if

$$\frac{\partial\phi(x, 0)}{\partial Y} = \theta_1(x) = \theta(x)H(1-x) + \theta_s(x)H(x-1), \quad x > 0 \quad (4.2)$$

and

$$\phi(x, 0) = f(x)H(1-x) - \frac{C}{2} H(x-1), \quad x > 0$$

where $f(x)$ and $\theta_s(x)$ are unknown functions, then the desired solution of Eq. (4.1) is

$$\begin{aligned} \phi(x, 0) = f(x) = & -\frac{C}{2} \frac{\Gamma(2/n+2)}{\Gamma^2(1/n+2)} \int_0^x \xi^{(1/n+2)-1} (1-\xi)^{(1/n+2)-1} d\xi \\ & + \frac{a^{-1}x^{1/n+2}}{\Gamma^2(1/n+2)} \int_x^1 (\xi-x)^{(1/n+2)-1} \xi^{-2/n+2} d\xi \int_0^{\xi} (\xi-\eta)^{(1/n+2)-1} \eta^{1/n+2} \theta(\eta) d\eta, \end{aligned} \quad (4.3)$$

when x is between 0 and 1.

To prove Eq. (4.3) the same procedure is followed as in Sec. 3, using the Mellin transform in place of the Laplace transform. Operating on both sides of (4.1) with the Mellin transform, defined by

$$M_s[f(x)] = \int_0^{\infty} x^{s-1} f(x) dx,$$

yields the relation³

$$aM_s[\phi(x, 0)] = \frac{\Gamma[1-s-(1/n+2)]\Gamma[s+(1/n+2)]}{\Gamma(1-s)\Gamma[s+(2/n+2)]} M_s[x^{2/n+2}\theta_1(x)]. \quad (4.4)$$

³This is the Mellin transform of a repeated fractional integral. See [10], vol. 2, p. 183.

A similar operation on Eq. (4.2) yields

$$\begin{aligned} M_s[\phi(x, 0)] &= M_s[f(x)H(1-x)] - \frac{C}{2} M_s[H(x-1)] = F(s)_+ + \frac{C}{2} \frac{1}{s_-}, \\ M_s[x^{2/n+2}\theta_1(x)] &= M_s[x^{2/n+2}\theta(x)H(1-x)] + M_s[x^{2/n+2}\theta_3(x)H(x-1)] \\ &= T(s)_+ + T_3(s)_-, \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} F(s)_+ &= \int_0^1 x^{s-1} f(x) dx, \\ \frac{1}{s_-} &= -M_s[H(x-1)], \quad \operatorname{Re}(s) < 0, \\ T(s)_+ &= \int_0^1 x^{s-1} x^{2/n+2} \theta(x) dx, \\ T_3(s)_- &= \int_1^\infty x^{s-1} x^{2/n+2} \theta_3(x) dx. \end{aligned}$$

Substitution of Eq. (4.5) into Eq. (4.4) yields

$$aF(s)_+ + a \frac{C}{2} \frac{1}{s_-} = \frac{\Gamma[1-s-(1/n+2)]}{\Gamma(1-s)} \frac{\Gamma[s+(1/n+2)]}{\Gamma[s+(2/n+2)]} [T(s)_+ + T_3(s)_-]. \quad (4.6)$$

As in Sec. 3, the regions of analyticity and the asymptotic behavior of F , T , T_3 are required and are shown as follows:

(i) $F(s)_+$ is an unknown function. It is assumed $f(x) = O(x^{1/n+2})$ as $x \rightarrow 0$; then $F(s)_+$ is analytic for $\operatorname{Re}(s) > -1/n+2$. In order to determine the asymptotic behavior of $F(s)_+$, substitute $x = \exp(-z)$ into the defining integral above, so that it takes the form of a Laplace integral. It is easily seen that if $f(x) = O(|1-x|^{-p})$, $p < 1$ as $x \rightarrow 1$ then $F(s)_+ = O(|s|^{-1+p})$ as $|s| \rightarrow \infty$. This assumption is necessary in order that $F(s)_+$ exist.

(ii) $T(s)_+$ is a known function. Since $\theta(x)$ is a known bounded function, $T(s)_+$ is analytic for $\operatorname{Re}(s) > -2/n+2$. The substitution $x = \exp(-z)$ shows that $T(s) = O(|s|^{-1})$ as $|s| \rightarrow \infty$.

(iii) $T_3(s)_-$ is an unknown function. By comparison with known potential solutions (i.e., potential flow about a flat plate with circulation) it is assumed that the unknown function $\theta_3(x) = O(x^{-2/n+2})$ as $x \rightarrow \infty$ so that $T_3(s)_-$ is analytic for $\operatorname{Re}(s) < 0$. If it is assumed that $\theta_3(x) = O(x-1)^{-q}$, $q < 1$ as $x \rightarrow 1$, the substitution $x = e^s$ obtains $T_3(s)_- = O(|s|^{-1+q})$ as $|s| \rightarrow \infty$.

(iv) The asymptotic behavior of ratios of gamma functions in Eq. (4.6) may be found by substituting $x = \exp(-\sigma)$ into the beta function representation,

$$\frac{\Gamma(s)}{\Gamma(s+\nu)} = \frac{1}{\Gamma(\nu)} \int_0^1 x^{s-1} (1-x)^{\nu-1} dx.$$

From this it is found that the function $\Gamma[s+(2/n+2)]/\Gamma[s+(1/n+2)]$ which is analytic for $\operatorname{Re}(s) < 1-(1/n+2)$ is of the order $|s|^{1/n+2}$ as $|s| \rightarrow \infty$, while $\Gamma[1-s-(1/n+2)]/\Gamma(1-s)$ is analytic for $\operatorname{Re}(s) < 1-(1/n+2)$ and is of the order $|s|^{-1/n+2}$ as $|s| \rightarrow \infty$.

Rearrangement of Eq. (4.6) gives

$$aF(s)_+ \frac{\Gamma[s + (2/n + 2)]}{\Gamma[s + (1/n + 2)]} = \frac{\Gamma[1 - s - (1/n + 2)]}{\Gamma(1 - s)} T_3(s)_- \\ - a \frac{C}{2} \frac{1}{s_-} \frac{\Gamma[s + (2/n + 2)]}{\Gamma[s + (1/n + 2)]} + T(s)_+ \frac{\Gamma[1 - s - (1/n + 2)]}{\Gamma(1 - s)}. \quad (4.7)$$

A little reflection on (i), (ii), (iii) and (iv) shows that all the functions occurring in Eq. (4.7) are analytic in the strip $-(1/n + 2) < \text{Re}(s) < 0$. As in Sec. 3, we can write

$$\frac{\Gamma[1 - s - (1/n + 2)]}{\Gamma(1 - s)} T(s)_+ = \frac{1}{2\pi i} \int_{\beta_2 - i\infty}^{\beta_2 + i\infty} \frac{1}{\omega - s} \frac{\Gamma[1 - \omega - (1/n + 2)]}{\Gamma(1 - \omega)} T(\omega)_+ d\omega \\ - \frac{1}{2\pi i} \int_{\beta_1 - i\infty}^{\beta_1 + i\infty} \frac{1}{\omega - s} \frac{\Gamma[1 - \omega - (1/n + 2)]}{\Gamma(1 - \omega)} T(\omega)_+ d\omega = \lambda_2(s)_- - \lambda_1(s)_+, \quad (4.8)$$

where $-(1/n + 2) < \beta_1 < \text{Re}(s) < \beta_2 < 0$ and $\lambda_2(s)_-$ and $\lambda_1(s)_+$ represent the first and the second integral on the right hand side. $\lambda_2(s)_-$ is analytic for $\text{Re}(s) < \beta_2$ while $\lambda_1(s)_+$ is analytic for $\text{Re}(s) > \beta_1$. Also, by subtracting the principal part of the singularity at $s = 0$

$$\frac{1}{s} \frac{\Gamma[s + (2/n + 2)]}{\Gamma[s + (1/n + 2)]} = \left\{ \frac{1}{s} \frac{\Gamma[s + (2/n + 2)]}{\Gamma[s + (1/n + 2)]} - \frac{1}{s} \frac{\Gamma(2/n + 2)}{\Gamma(1/n + 2)} \right\}_+ + \frac{1}{s_-} \frac{\Gamma(2/n + 2)}{\Gamma(1/n + 2)}. \quad (4.9)$$

The bracketed term on the right is analytic for $\text{Re}(s) > -2/n + 2$, while the second term on the right is analytic for $\text{Re}(s) < 0$. Introducing the results of Eq. (4.8) and Eq. (4.9) into Eq. (4.7) gives

$$E_1(s) = aF(s)_+ \frac{\Gamma[s + (2/n + 2)]}{\Gamma[s + (1/n + 2)]} + \lambda_1(s)_+ \\ + a \frac{C}{2} \left\{ \frac{1}{s} \frac{\Gamma[s + (2/n + 2)]}{\Gamma[s + (1/n + 2)]} - \frac{1}{s} \frac{\Gamma(2/n + 2)}{\Gamma(1/n + 2)} \right\}_+ \quad (4.10) \\ = \frac{\Gamma[1 - s - (1/n + 2)]}{\Gamma(1 - s)} T_3(s)_- + \lambda_2(s)_- - a \frac{C}{2} \frac{\Gamma(2/n + 2)}{\Gamma(1/n + 2)} \frac{1}{s_-}.$$

As in Sec. 3, since the left and right sides of this equation are analytic in overlapping half planes, an entire function $E_1(s)$ can be defined. From the estimates in (i), (ii), (iii) and (iv) it is found that

$$E_1(s)_+ = 0 \quad |s|^{(1/n+2)-1+p} \quad (4.11)$$

and

$$E_1(s)_- = 0 \quad |s|^{-(1/n+2)-1+q}$$

as

$$|s| \rightarrow \infty, \quad \text{where } p, q < 1.$$

These estimates imply $E_1(s) \equiv 0$. This determines $F(s)_+$ and $T_3(s)_-$ in Eq. (4.10) as follows

$$F(s)_+ = -a^{-1} \frac{\Gamma[s + (1/n + 2)]}{\Gamma[s + (2/n + 2)]} \lambda_1(s)_+ \quad (4.12) \\ - \frac{C}{2} \left\{ \frac{1}{s} - \frac{1}{s} \frac{\Gamma(2/n + 2)}{\Gamma(1/n + 2)} \frac{\Gamma[s + (1/n + 2)]}{\Gamma[s + (2/n + 2)]} \right\}_+ \quad \text{Re}(s) > -1/n + 2$$

$$T_2(s) = -\frac{\Gamma(1-s)}{\Gamma[1-s-(1/n+2)]} \lambda_2(s) - \quad (4.13)$$

$$+ a \frac{C}{2} \frac{\Gamma(2/n+2)}{\Gamma(1/n+2)} \frac{1}{s} \frac{\Gamma(1-s)}{\Gamma[1-s-(1/n+2)]} \quad \text{Re}(s) < 0.$$

The next step is to take the inverse Mellin transform of Eq. (4.12) using the Mellin convolution in the same way that we used the Laplace convolution in Sec. 3. The result is Eq. (4.3) which was to be demonstrated.

5. Solution of the mixed boundary value problem stated in Sec. 1. The original problem to be solved was a mixed boundary value problem. Since from Sec. 4, $\phi(x, 0)$ is known on the interval $(0, 1)$, we know $\phi(x, 0)$ for the infinite interval. This reduces the mixed boundary value problem to a much simpler Dirichlet problem.

$$Y^n \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial Y^2} = 0, \quad Y \geq 0, \quad n \geq 0; \quad (5.1)$$

$$\phi(x, 0) = \phi_0(x), \quad -\infty < x < \infty;$$

$$\phi(x, \infty) = 0.$$

Here

$$\phi_0(x) = \begin{cases} 0 & x < 0 \\ f(x) & 0 < x < 1 \\ -C/2 & x > 1 \end{cases}$$

where $f(x)$ is defined by Eq. (4.3). Use of Laplace transforms as in Eq. (3.6) yields the bounded solution shown in Eq. (3.7). From Eq. (3.9)

$$\Phi(s, 0) = \frac{\pi}{2} \frac{(-s)^{-1/2n+4}}{(1/n+2)^{1/n+2} \sin(\pi/n+2) \Gamma(n+1/n+2)} A = \Phi_0(s), \quad (5.2)$$

where $\Phi_0(s)$ is the two-sided Laplace integral of $\phi_0(x)$. Solving Eq. (5.2) for A and substituting it into Eq. (3.7),

$$\Phi = \frac{2}{\pi} (1/n+2)^{1/n+2} \sin(\pi/n+2) \Gamma(n+1/n+2) (-s^2)^{1/2n+4} \cdot Y^{1/2} K_{1/n+2} \left[(-s^2)^{1/2} \frac{2}{n+2} Y^{n+2/2} \right] \Phi_0 \quad \text{Re}(s) = 0. \quad (5.3)$$

The inversion of this gives the result

$$\phi(x, Y) = d_n \int_{-\infty}^{\infty} \frac{\phi_0(\xi) Y d\xi}{[(x-\xi)^2 + (2/n+2) Y^{n+2}]^{(1/n+2)+1/2}},$$

where $d_n = \pi^{-3/2} (2/n+2)^{2/n+2} \Gamma[(1/n+2)+1/2] \Gamma(n+1/n+2) \sin(\pi/n+2)$. Since this result will not be used in Part II, no further comment will be made.

PART II. AIRFOIL IN A SONIC SHEAR FLOW JET

6. Solution of shear flow problem. As seen in Sec. 1, for a symmetrical Mach

number distribution the potential ϕ must satisfy

$$(1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{2}{M} \frac{dM}{dy} \frac{\partial \phi}{\partial y} = 0 \quad (6.1)$$

for $y \geq 0$, and the boundary conditions

$$\begin{aligned} \phi(x, 0) &= 0 & x < 0, \\ &= -C_{1/2} & x > 1, \\ \frac{\partial \phi(x, 0)}{\partial y} &= -2\alpha_0(x) & 0 < x < 1, \\ \phi(x, \infty) &= 0. \end{aligned} \quad (6.2)$$

Introducing the new independent variable

$$Y = b \int_0^y M^2(y) dy, \quad (6.3)$$

where b is a constant, Eq. (7.1) is transformed into

$$\frac{1 - M^2}{b^2 M^4} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial Y^2} = 0. \quad (6.4)$$

Let $M(y)$, an arbitrary function until now, be defined by

$$\frac{1 - M^2}{b^2 M^4} = Y^n = \left(b \int_0^y M^2 dy \right)^n, \quad (6.5)$$

where n is a non-negative number, so that Eq. (6.4) becomes the generalized Tricomi equation treated in Part I. The solution of (6.5), which gives the Mach number distribution, will be investigated in Sec. 7. The transformation by Eq. (6.3) also changes the third equation in the boundary conditions, Eq. (6.2) to

$$\frac{\partial \phi(x, 0)}{\partial Y} = -\frac{2\alpha_0(x)}{bM_0^2}, \quad 0 < x < 1. \quad (6.6)$$

It is seen immediately that this is the problem solved in Part I with the notational change $\theta(x) = -2\delta_0(x)/bM_0^2$ and $C = C_i$. The solution of this problem involves the constant C_i which has been presumed specified in advance. To determine C_i the Kutta-Joukowski condition should be used. This condition states that the flow must leave the trailing edge smoothly. In a linearized approach, this means that the streamline leaving the trailing edge of the airfoil should have finite slope at the trailing edge. In Appendix II it is shown that the slope is finite provided that

$$C_i = \frac{2\Gamma(1/n + 2)}{\Gamma(2/n + 2)} \frac{(1/n + 2)^{n/n+2}}{\Gamma(n + 1/n + 2)} \int_0^1 (1 - \eta)^{(1/n+2)-1} \eta^{1/n+2} [-2\alpha_0(\eta)/bM_0^2] d\eta. \quad (6.7)$$

This gives the lift coefficient in terms of the airfoil slope. The pressure coefficient on the airfoil can be calculated by taking the x derivative of Eq. (4.3), since $C_p(x, 0) = \partial \phi(x, 0)/\partial x$. This may be accomplished more easily if the integration variable in the second integral in Eq. (4.3) is changed by the substitution $z = x/\xi$. The result is

$$C_p(x, 0) = e_n \int_1^x (1-z)^{(1/n+2)^{-1}-1} z^{(1/n+2)^{-1}-1} dz \frac{\partial}{\partial x} \quad (6.8)$$

$$\cdot \int_0^{x/z} \left(\frac{x}{z} - \eta \right)^{(1/n+2)^{-1}-1} \eta^{1/n+2} [-2\alpha_0(\eta)/bM_0^2] d\eta$$

where

$$e_n = \frac{-(1/n+2)^{n/n+2}}{\Gamma(n+1/n+2)\Gamma(1/n+2)}.$$

It will be shown in Sec. 8 that Eq. (6.8) can be integrated with a simple result if $\alpha_0(x)$ is a polynomial.

7. Mach number profile. Certain properties of the upstream Mach number distribution, $M(y)$, which are defined by the solution of Eq. (6.5), can easily be seen without completely solving this equation. These will be enumerated below.

(i) If $n = 0$, $M = M_0$ is constant and less than one. In this case $b(n = 0) = b_0$ can be expressed in terms of M_0 as $b_0 = (1 - M_0^2)^{1/2}/M_0^2$. Noticing how b_0 occurs in Eq. (6.8), it is seen that b_0 is essentially the Prandtl-Glauert correction factor for incompressible flow.

(ii) For $n = 0$, $M \rightarrow 1$ as $y \rightarrow 0$ while $M \rightarrow 0$ as $y \rightarrow \infty$.

(iii) To see how M behaves near $y = 0$, rewrite Eq. (6.5) as

$$b^{(n+2)/n} \int_0^y M^2 dy = \left(\frac{1 - M^2}{M^4} \right)^{1/n} \quad (7.1)$$

and substitute a series, $M = 1 - y^\nu (a_0 + a_1 y + \dots)$, for M . Solving the resulting equation for a_0 and ν it is found that

$$M = 1 - \frac{1}{2} b^{n+2} y^n + \dots \quad (7.2)$$

This shows that in addition to being sonic at $y = 0$, the profile becomes increasingly flat near $y = 0$ as n increases. This is of some interest since, as is well known, purely sonic uniform flow cannot be linearized. Notice also that $b \rightarrow 0$ implies uniform sonic flow, while from Eq. (6.8) the pressure becomes infinite as $b \rightarrow 0$, so that the linearization has become invalid.

(iv) The behavior for small n can also be seen from Eq. (7.2). If $0 < n < 1$, M has a cusp at $y = 0$ which becomes sharper as n decreases. Upon taking the limit as $n \rightarrow 0$ the cusp collapses on itself leaving uniform subsonic flow.

Since the solution of Eq. (7.1) for M as a function of y is difficult to obtain explicitly, the inverse of this function $y = y(M)$ will be considered. Denoting M^2 by ξ and differentiating Eq. (7.1) with respect to y we get

$$b^{(n+2)/n} \xi = \frac{d}{d\xi} \left(\frac{1 - \xi}{\xi^2} \right)^{1/n} \frac{d\xi}{dy}.$$

Upon separating variables and integrating,

$$b^{(n+2)/n} y = \int_1^{M^2} \frac{1}{\xi} \frac{d}{d\xi} \left(\frac{1 - \xi}{\xi^2} \right)^{1/n} d\xi$$

is obtained. An integration by parts gives

$$b^{(n+2)/n} y = \frac{1}{M^2} \left(\frac{1 - M^2}{M^4} \right)^{1/n} - \int_{M^2}^1 \frac{1}{\xi^2} \left(\frac{1 - \xi}{\xi^2} \right)^{1/n} d\xi. \quad (7.3)$$

This is the desired inverse function. If n equals 1 or 2 the integration can be carried out explicitly. For large n a series in $1/n$ is appropriate. The results of these calculations are given in Fig. 2. In this figure M is plotted versus $y b^{(n+2)/n}$. If $b = 1$ the various curves are in the correct proportions as plotted, however, if $b \neq 1$ the scale changes for each curve. If n is very large the profiles are again approximately in the proper proportions. Note that as $n \rightarrow \infty$ the profile approaches a uniform sonic jet of width $2/b$; twice the reciprocal of b can be considered roughly to be an effective jet width.

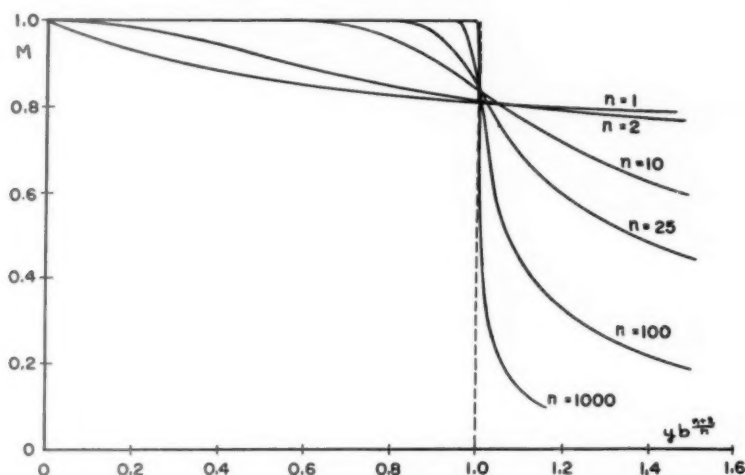


FIG. 2. Mach number versus vertical distance with n as parameter.

8. Calculations of C_p , C_l and $C.P.$ In this section the calculation of the pressure coefficient, lift coefficient and center of pressure ($C.P.$) will be carried out for the case where $\alpha_0(x)$ is a polynomial. In this case $-2\alpha_0(x)/bM_0^2$ can be expressed as a sum of terms like $a_i x^i$ and the lift coefficient, Eq. (6.7) will be a sum of terms like

$$C_{li} = 2a_i \frac{\Gamma(1/n + 2)(1/n + 2)^{n/n+2}}{\Gamma(2/n + 2)\Gamma(n + 1/n + 2)} \int_0^1 (1 - \eta)^{(1/n+2)-1} \eta^{i+(1/n+2)} d\eta.$$

This integral is a beta function which is easily expressed as

$$C_{li} = 2a_i \frac{\Gamma(1/n + 2)(1/n + 2)^{n/n+2}}{\Gamma(2/n + 2)\Gamma(n + 1/n + 2)} \frac{\Gamma(1/n + 2)\Gamma[(1/n + 2) + j + 1]}{\Gamma[(2/n + 2) + j + 1]}. \quad (8.1)$$

The pressure coefficient, Eq. (6.8), on the airfoil will be a sum of terms like

$$C_{pi} = g_n \int_z^1 (1 - z)^{(1/n+2)-1} z^{(1/n+2)-1} dz \frac{\partial}{\partial x} \int_0^{x/z} \left(\frac{x}{z} - \eta\right)^{(1/n+2)-1} \eta^{i+(1/n+2)} d\eta,$$

where

$$g_n = -\frac{a_i(1/n + 2)^{n/n+2}}{\Gamma(n + 1/n + 2)\Gamma(1/n + 2)}.$$

The inner integral, a fractional integral of the Riemann-Liouville type can be evaluated

as

$$\int_0^{x/z} \left(\frac{x}{z} - \eta \right)^{(1/n+2)^{-1}} \eta^{j+(1/n+2)} d\eta = \Gamma(1/n+2) \frac{\Gamma[j+1+(1/n+2)]}{\Gamma[j+1+(2/n+2)]} (x/z)^{j+(2/n+2)}.$$

Then

$$C_{pi} = k_n \int_x^1 (1-z)^{(1/n+2)^{-1}-j-1} z^{-(1/n+2)^{-1}-j-1} dz,$$

where

$$k_n = -\frac{a_i(1/n+2)^{n/n+2}}{\Gamma(n+1/n+2)} \frac{\Gamma[j+1+(1/n+2)]}{\Gamma[j+1+(2/n+2)]} x^{j-1+(2/n+2)}.$$

By a change of the variable of integration, $t = (1-z)/z$ this becomes

$$C_{pi} = -a_i \frac{(1/n+2)^{n/n+2}}{\Gamma(n+1/n+2)} \frac{\Gamma[j+1+(1/n+2)]}{\Gamma[j+1+(2/n+2)]} x^{j-1+(2/n+2)} \cdot \int_0^{(1-x)/x} t^{(1/n+2)^{-1}} (1+t)^j dt. \quad (8.2)$$

The integration here can be easily carried out for any value of j by expanding $(1+t)^j$.

Let a moment coefficient be defined by

$$C_m = -2 \int_0^1 x C_p dx. \quad (8.3)$$

This can be computed after C_p is known. A more useful quantity which will be used is the center of pressure defined by

$$C.P. = C_m/C_i. \quad (8.4)$$

These quantities (C_i , C_p , $C.P.$) are calculated below for the special cases of flat plate and circular arc airfoils.

(i) *Flat plate airfoil at angle of attack δ .*

In this case, using the above notation, $\alpha_0(x) = -\delta$ is a constant. Then

$$\frac{-2\alpha_0(x)}{bM_0^2} = \frac{2\delta}{bM_0^2} = a_0.$$

From Eq. (8.1),

$$C_i = \frac{2\delta}{bM_0^2} \left(\frac{1}{n+2} \right)^{n/n+2} \frac{\Gamma^3(1/n+2)}{\Gamma^2(2/n+2)\Gamma(n+1/n+2)}. \quad (8.5)$$

If $n = 0$ (uniform subsonic flow) this gives the familiar result

$$C_i = \frac{2\pi\delta}{bM_0^2} = \frac{2\pi\delta}{(1-M^2)^{1/2}},$$

where $b = (1-M^2)^{1/2}/M^2$, M constant. If $n \rightarrow \infty$ (sonic jet) the result is $C_i = 8\delta/b = 4h\delta$. Here the jet thickness $h = 2/b$ has been used. Equation (8.5) is plotted in Fig. 3. From Eq. (8.2) the pressure coefficient

$$C_p/C_i = -\frac{\Gamma(2/n+2)}{\Gamma^2(1/n+2)} x^{(1/n+2)^{-1}} (1-x)^{1/n+2} \quad (8.6)$$

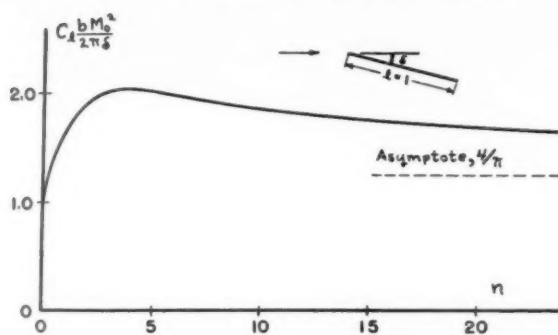
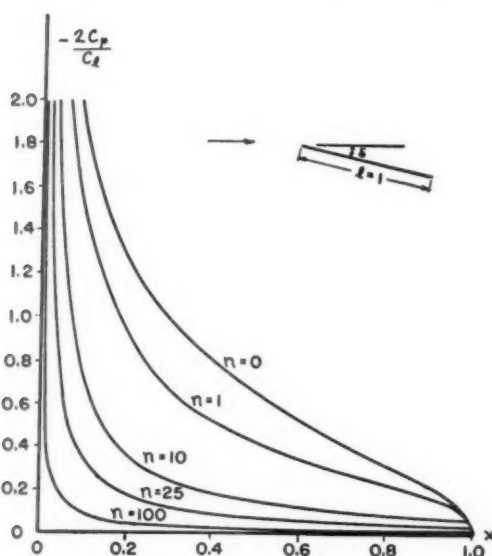
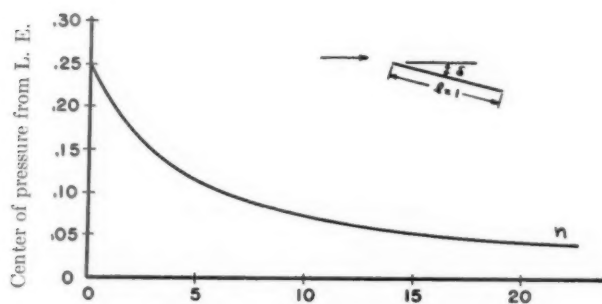
FIG. 3. Lift coefficient on flat plate versus n .

FIG. 4. Pressure coefficient on flat plate airfoil versus distance along airfoil.

FIG. 5. Center of pressure on flat plate airfoil versus n .

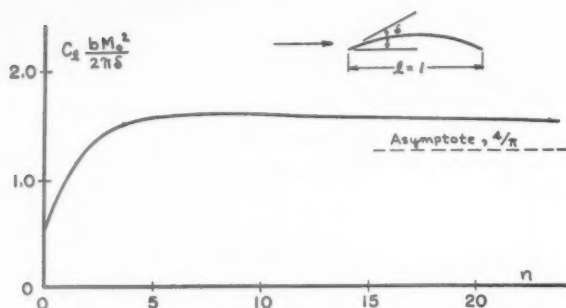
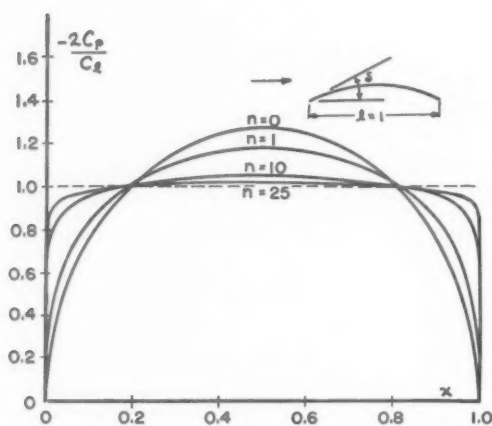
FIG. 6. Lift coefficient on circular arc airfoil versus n .

FIG. 7. Pressure coefficient on circular arc airfoil versus distance along airfoil.

is obtained. In Fig. 4, $-2 C_p/C_l$ has been plotted versus x so that the "area" under each of the curves is unity. Notice that for $n = 0$ the leading edge singularity is like $x^{-1/2}$. As n becomes larger this singularity becomes stronger, approaching x^{-1} as $n \rightarrow \infty$. The center of pressure is obtained by substituting Eq. (8.6) into Eq. (8.3). The result is the simple expression

$$C.P. = \frac{1}{n + 4}. \quad (8.7)$$

This shows that for $n = 0$ the center of pressure is at the quarter chord and approaches the leading edge as n increases. This is presented in Fig. 5.

(ii) *Circular arc airfoil at zero geometric angle of attack.* For the circular arc $\alpha_0(x) = \delta(1 - 2x)$ where δ is the angle which the leading edge makes with the free stream. Then

$$\frac{-2\alpha_0(x)}{bM_0^2} = a_0 + a_1x, \quad a_0 = \frac{-2\delta}{bM_0^2}, \quad a_1 = \frac{4\delta}{bM_0^2}.$$

The lift coefficient and pressure coefficient are easily calculated from Eqs. (8.1) and

(8.2). We obtain

$$C_l = \left(\frac{2\delta}{bM_0^2} \right) (n + 2/n + 4)(1/n + 2)^{n/n+2} \frac{\Gamma^3(1/n + 2)}{\Gamma^{2/2/n + 2} \Gamma(n + 1/n + 2)}, \quad (8.8)$$

$$\frac{C_p}{C_l} = -\frac{\Gamma(2/n + 2)}{\Gamma^2(1/n + 2)} (n + 4)x^{1/n+2}(1-x)^{1/n+2}. \quad (8.9)$$

The center of pressure is at $x = .5$ for all n as can be seen from the symmetry of the pressure coefficient. Equations (8.8) and (8.9) are presented in Figs. 6 and 7. Note in Fig. 7 that as $n \rightarrow \infty$ the pressure coefficient becomes constant along the airfoil.

9. Discussion. It has been shown that for a particular class of Mach number profiles which are characterized by a sonic line, the linearized compressible shear flow equation can be transformed into the generalized Tricomi equation. The boundary value problem, which describes the perturbation of the main stream flow by an arbitrary two-dimensional camber surface, has been formulated and solved. The question of agreement with reality remains.

It is probable that for large values of n the linearization breaks down unless restrictions are made on the parameter b , which should be large as n becomes large. That is, the effective jet width, $2/b$, must be much less than the chord length. This conjecture is supported by the following observation; by the foregoing theory the lift on an airfoil in the limiting case $n \rightarrow \infty$ is the same as that calculated by a simple momentum exchange brought about by the sonic jet turning through an angle equal to the trailing edge angle of the airfoil. Such a situation can only be realized if the jet width is much less than the chord length.

APPENDIX I.

Jump condition. Let C denote the curve indicated in Fig. 8, consisting of two straight lines and a circle of radius R . If $\nabla\phi$ has continuous derivatives inside C we can write

$$\int_C \left(\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy \right) = 0$$

or

$$\int_0^R [C_p(x, 0_+) - C_p(x, 0_-)] dx + \int_0^{2\pi} \frac{\partial\phi}{\partial\theta} d\theta = 0.$$

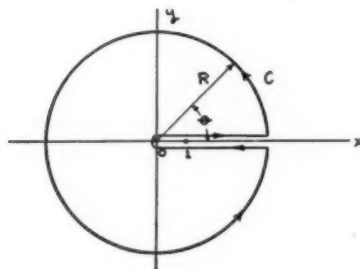


FIG. 8.

Here $\partial\phi/\partial x = C_p$ has been used. C_p must be continuous except across the airfoil itself, therefore $C_p(x, 0_+) - C_p(x, 0_-)$ must be zero for $x > 1$. Then the first integral is the negative of the lift coefficient,

$$C_l = \int_0^1 [C_p(x, 0_-) - C_p(x, 0_+)] dx = \frac{\text{Lift}}{(\gamma/2)PM_0^2},$$

while the second integral is $\phi(R, 0_-) - \phi(R, 0_+)$. Hence

$$C_l = \phi(R, 0_-) - \phi(R, 0_+).$$

Since C_l is independent of R the jump across the x -axis must be constant.

APPENDIX II.

Kutta-Joukowski condition. It is necessary that $\partial\phi(x, 0)/\partial Y$ be bounded as $x \rightarrow 1$ from the right. That is, $\theta_3(x)$ in Eq. (4.2) should be bounded. This condition will be satisfied only for a certain value of C . $\theta_3(x)$ can be found by inverting Eq. (4.13). Proceeding as with the inversion of Eq. (4.12) yields, after an integration by parts.

$$\begin{aligned} \theta_3(x) = & \frac{x^{-1/n+2}(x-1)^{-1/n+2}}{\Gamma(n+1/n+2)\Gamma(1/n+2)} \left[-\frac{aC}{2} \Gamma(2/n+2) \right. \\ & \left. - \int_0^1 (1-\eta)^{(1/n+2)-1} \eta^{1/n+2} \theta(\eta) d\eta \right] + \frac{(n+1/n+2)x^{-1/n+2}}{\Gamma(n+1/n+2)\Gamma(1/n+2)} \\ & \cdot \int_1^x \xi^{2/n+2}(x-\xi)^{-1/n+2} d\xi \int_0^1 (\xi-\eta)^{(1/n+2)-2} \eta^{1/n+2} \theta(\eta) d\eta. \end{aligned}$$

Therefore $\theta_3(x)$ will be bounded at $x = 1$ if the coefficient of $(x-1)^{-1/n+2}$ is zero, that is, if

$$\frac{C}{2} = -\frac{a^{-1}}{\Gamma(2/n+2)} \int_0^1 (1-\eta)^{(1/n+2)-1} \eta^{1/n+2} \theta(\eta) d\eta.$$

Using the definition of a , Eq. (3.3), gives

$$\frac{C}{2} = (1/n+2)^{n/n+2} \frac{\Gamma(1/n+2)}{\Gamma(2/n+2)\Gamma(n+1/n+2)} \int_0^1 (1-\eta)^{(1/n+2)-1} \eta^{1/n+2} \theta(\eta) d\eta.$$

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ON THE APPLICATION OF ELLIPTIC FUNCTIONS IN NON-LINEAR FORCED OSCILLATIONS*

BY

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1. Introduction. A system having an equation of motion of the Duffing type is known to respond in free vibrations in the form of elliptic functions, and the period is expressible in terms of complete elliptic integrals. It seems appropriate, therefore, to analyze the response of such systems when they are subject to periodic forcing functions which are themselves elliptic functions. It is the purpose of this paper to examine this problem in detail.

We shall show that there exist quite general cases where exact single-term and multiple-term elliptic responses can arise under forcing functions consisting either of a single elliptic function, or of a combination of these. It will be shown that all known results of the perturbation or iteration method are simply special cases of these exact solutions, and the former emerge easily from the present analysis if one expands the results of this paper in terms of the moduli of the elliptic functions and retains only terms which are linear in the second power of the modulus. Higher order approximations may easily be found from the multiple-term exact solutions by comparing orders of magnitude of the various components in an appropriate fashion. Finally, we shall demonstrate exact subharmonic solutions which are elliptic functions and which may, or may not, be simple. The simple elliptic subharmonic solutions reduce to the simple subharmonics defined by Rosenberg [6]** if one sets the modulus equal to zero.

In view of the fact that there is a fair amount of uniformity concerning the notation used in connection with elliptic functions, we have adopted the commonly accepted notation of Whittaker and Watson [9].

2. Non-linear free vibrations. The equation of free vibrations of a Duffing system is

$$m \frac{d^2 x}{dt^2} + k_{s1}x + k_{s3}x^3 = 0. \quad (1)$$

Introducing $\tau = t(k_{s1}/m)^{1/2}$ and $\beta = k_{s3}/k_{s1}$, we obtain a new equation of motion

$$\frac{d^2 x}{d\tau^2} + x + \beta x^3 = 0. \quad (2)$$

Let us assume that the peak amplitude of the oscillation x is equal to A , and it occurs at $\tau = 0$. Then it is well known [9] that the solution of (2) is given by the Jacobian elliptic function $\text{cn } u$. The exact form of the solution is

$$x = A \text{ cn } (p\tau, k), \quad p = (1 + \beta A^2)^{1/2}, \quad k = [\beta A^2 / 2(1 + \beta A^2)]^{1/2}, \quad (3)$$

where k is the modulus of the elliptic function and p may be taken as the "circular frequency."

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**Numbers in square brackets refer to the bibliography at the end of the paper.

If we have a soft non-linear spring, i.e., β is negative, the solution may be transformed [1] into the following form in order to have the modulus remaining in the usual range from 0 to 1, [2, 5],

$$x = A \operatorname{sn} [p_1 \tau + K(k_1), k_1], \quad p_1 = (1 + \beta A^2/2)^{\frac{1}{2}}, \quad k_1 = [-\beta A^2/(2 + \beta A^2)]^{\frac{1}{2}}. \quad (4)$$

Here $K(k)$ is the complete elliptic integral of the first kind.

Similarly, when $\beta A^2 < -1$, the solution can be put in the following form.

$$x = A \operatorname{ns} [p_2 \tau + K(k_2), k_2], \quad p_2 = (-\beta A^2/2)^{1/2}, \quad k_2 = [-(2 + \beta A^2)/\beta A^2]^{1/2}. \quad (5)$$

We note that in this case the solution is no longer bounded. The meaning is quite obvious physically. When β is negative and A is so large that $\beta A^2 < -1$, the total spring force—i.e., the linear and the non-linear part together—becomes negative. Under this circumstance the mass will move away from the origin $x = 0$ indefinitely. As a matter of fact, the quantity A should be interpreted here as the amplitude at $\tau = 0$, not as the peak amplitude.

Both the functions cnu and snu have a period in u equal to $4K$. The complete elliptic integral of the first kind, K , is then the quarter period of the free oscillation. The period in τ is given by

$$T_\tau = 4K/p. \quad (6)$$

For the case of a hard spring it is

$$T_\tau = 4(1 - 2k^2)^{1/2} K(k). \quad (7)$$

For a system with a soft spring it is

$$T_\tau = 4(1 + k_1^2)^{1/2} K(k_1). \quad (8)$$

Figures 1 and 2 indicate these relationships between βA^2 , k^2 , k_1^2 and T_τ in graphic form.

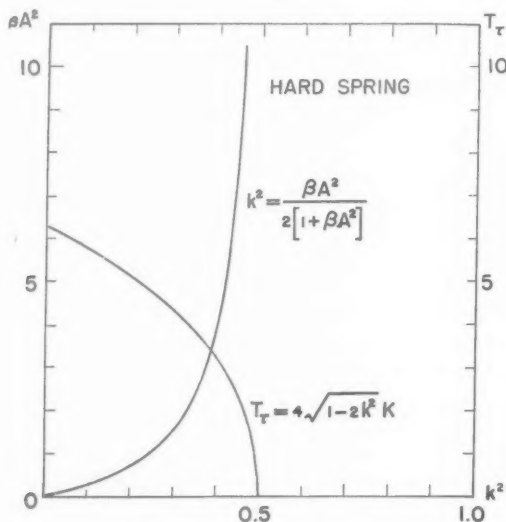


Fig. 1. Free vibration period and amplitude as functions of the modulus for a system with a hard spring.

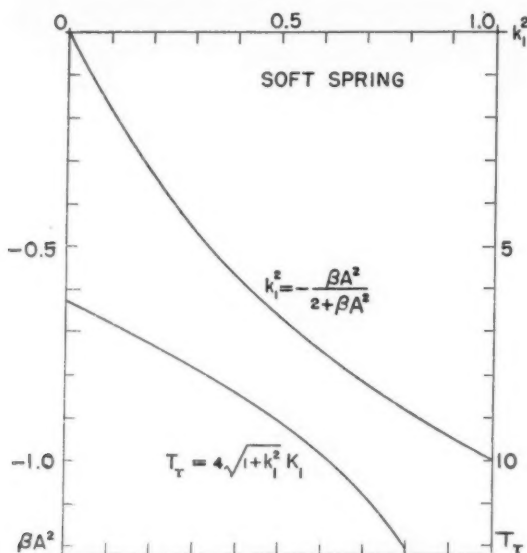


Fig. 2. Free vibration period and amplitude as functions of the modulus for a system with a soft spring.

When the non-linearity and the peak amplitude are such that $|\beta A^2|$ is small in comparison with unity, the modulus k will also be small. Expanding the quarter period K in powers of k^2 and substituting the expansion into (7) we find for the period

$$T_\tau = 2\pi[1 - \frac{3}{4}k^2 + O(k^4)]. \quad (9)$$

The equivalent circular frequency p_{er} can be easily found as,

$$p_{er}^2 = (2\pi/T_\tau)^2 = 1 + \frac{3}{4}\beta A^2 + O(\beta^2 A^4). \quad (10)$$

This agrees with the result obtained by the perturbation method when $|\beta|$ is assumed small [8].

3. Some exact solutions of forced vibration. In this section we shall examine the response of a Duffing system to forced vibration. We assume that small quantities of positive damping are present in the physical system, but absent from the equation of motion. Consequently the periodic solutions of interest are those usually referred to as steady state vibrations. We shall assume here that the effect of small damping on the motion is slight.

Let us consider a system whose equation of motion is

$$\frac{d^2x}{d\tau^2} + x + \beta x^3 = F(\tau), \quad (11)$$

where $F(\tau)$ is the external forcing function divided by k_{s1} . The forcing function $F(\tau)$ in (11) is usually taken to be a simple harmonic function of an arbitrary circular frequency. Here, however, this is not the case. Instead we shall try to find out what type of external forcing function will excite a steady state response, which is an exact solution of the equation of motion, and which is expressible in a simple form.

3.1 Simple elliptic response. First let us ask under what condition the response x as a function of τ will be proportional to the forcing function, leaving the precise form of the function undetermined. Putting $F(\tau) = Bx$, the equation of motion becomes

$$\frac{d^2x}{d\tau^2} + (1 - B)x + \beta x^3 = 0. \quad (12)$$

The solution of this equation can be readily found to be

$$x = A \operatorname{cn}(p\tau, k), \quad (13)$$

where

$$p = (1 - B + \beta A^2)^{1/2}, \quad k = [\beta A^2 / 2(1 - B + \beta A^2)]^{1/2}. \quad (13a)$$

We note that there are four parameters, A , B , p , and k involved in the solution and two relationships (13a) between them. Hence, any two of them may be arbitrarily chosen and the other two will be determined by (13a). Such a set of parameters will assure a response like (13). For example, if we take p and k as given we have

$$A = (2/\beta)^{1/2}pk, \quad B = 1 - p^2(1 - 2k^2). \quad (14)$$

The response x and the necessary forcing function $F(\tau)$ are respectively

$$x = A \operatorname{cn}(p\tau, k) = (2/\beta)^{1/2}pk \operatorname{cn}(p\tau, k), \quad (15)$$

$$F(\tau) = BA \operatorname{cn}(p\tau, k) = (2/\beta)^{1/2}pk[1 - p^2(1 - 2k^2)] \operatorname{cn}(p\tau, k). \quad (16)$$

The parameters p and k in $\operatorname{cn}(p\tau, k)$ play important geometrical roles. The modulus k controls the proportions of the higher harmonics in the Fourier analysis while p and k together determine the periodicity of the function. The period in τ is given by (6). Figure 3 shows curves of T_r as functions of p and k .

A response consisting of only a single term of elliptic function like (13) will be called a *simple elliptic response*. It should be emphasized here that not every elliptic forcing excites a simple elliptic response. The amplitude of the elliptic forcing function must bear a definite relationship to the parameters p and k . We shall denote those forcing functions satisfying this relationship as *avored* elliptic forcing functions.

In case of a soft non-linear spring, $\beta < 0$, the solution of (12) is

$$x = A \operatorname{sn}(p_1\tau, k_1),$$

where

$$p_1 = (1 - B + \beta A^2/2)^{1/2}, \quad k_1 = [-\beta A^2/2(1 - B + \beta A^2/2)]^{1/2}. \quad (17)$$

If we consider p_1 and k_1 as given we must have

$$B = 1 - p_1^2(1 + k_1^2), \quad A = (-2/\beta)^{1/2}p_1k_1 \quad (18)$$

and the forcing function

$$F(\tau) = BA \operatorname{sn}(p_1\tau, k_1) = (-2/\beta)^{1/2}p_1k_1[1 - p_1^2(1 + k_1^2)] \operatorname{sn}(p_1\tau, k_1). \quad (19)$$

Equations (17) may also be obtained through the application of the first order transformation mentioned in Sec. 2. From here on we shall not, therefore, treat the cases

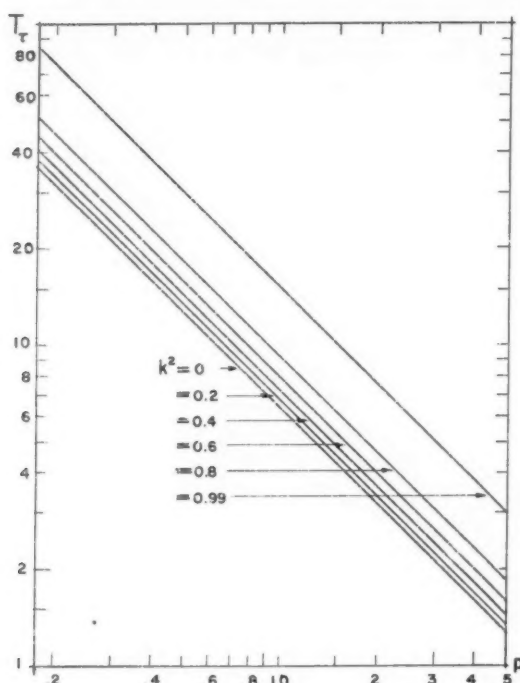


FIG. 3. Period in τ of the elliptic functions $cn(pr, k)$ and $sn(pr, k)$.

with soft springs separately. Whenever the modulus k appears outside the standard region of 0 to 1, one of the first order transformations must be used.*

The quantity $1/B$ is analogous to the well-known magnification factor—i.e., the ratio of response amplitude to forcing amplitude. In the linear case it gives the response amplitude for a forcing amplitude of unity. Near the point where $1/B$ approaches infinity resonance occurs. With non-linear systems it is better to take a somewhat different view. In linear cases it is customary to consider the forcing amplitude as given and the problem is to find the response amplitudes. In the non-linear case the response amplitude is intimately connected with the period, which is determined by p and k , and the wave form, which is determined by k . It is therefore desirable to consider the response amplitude as known and the problem is to find the necessary forcing amplitude. Hence, instead of $1/B$ we shall discuss the factor B , which is the ratio of forcing amplitude to response amplitude and will be called the *forcing amplitude factor*. Figures 4 and 5 show the curves of B vs. p with k^2 as a parameter for hard and soft non-linear springs respectively.

In Figs. 4 and 5 the curves of $k^2 = 0$ correspond to the usual linear case. B is given by $(1 - p^2)$ or the magnification factor $1/B = 1/(1 - p^2)$. Each of the curves in Fig.

*In a slightly different form, the simple elliptic response under one term elliptic forcing discussed here was treated by Helfenstein [3]. The material presented here was retained nevertheless because it is basic to all further development, and because Helfenstein's paper is not easily accessible.

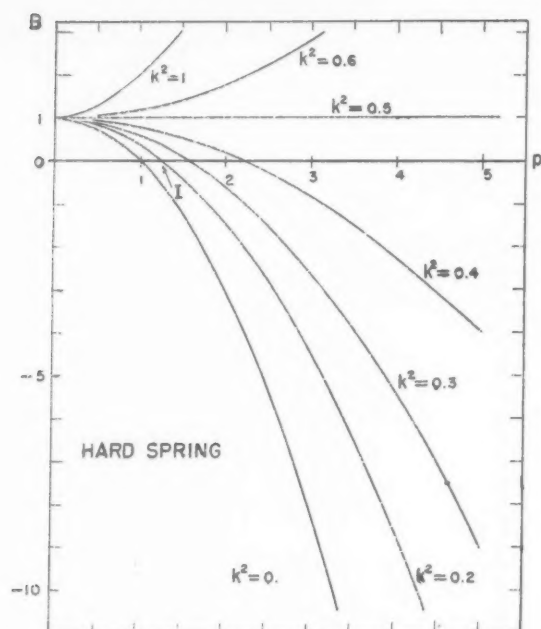


FIG. 4. Forcing amplitude factor B curves of simple elliptic response for a hard spring system.

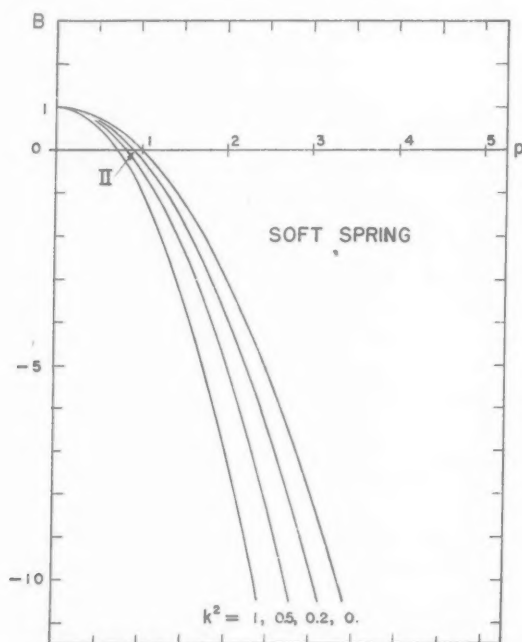


FIG. 5. Forcing amplitude factor B curves of simple elliptic response for a soft spring system.

5 and those in Fig. 4 with $k^2 < 0.5$ cross the p -axis once. Near the points of crossing, such as the points *I* and *II*, the value of B is extremely small. However, the response amplitude A is given by (14) and is finite. This means that near the crossing points extremely small external forcing amplitudes are capable of maintaining the response. This is the phenomenon of resonance. At any of the crossing points the external forcing required is zero. Here we have the case of free vibration. Thus, in non-linear systems such as those under discussion, the free vibrations emerge quite naturally as special cases of forced vibrations.

3.2 Two-term elliptic forcing. Results similar to those obtained in Sec. 3.1 can be readily found for other cases. For example, let us require that the forcing function $F(\tau)$ be $B_1x + B_3x^3$. The equation of motion may then be put as,

$$\frac{d^2x}{d\tau^2} + (1 - B_1)x + (\beta - B_3)x^3 = 0, \quad (20)$$

and its solution can be easily found. It is

$$x = A \operatorname{cn}(p\tau, k), \quad (21)$$

where

$$p = [1 - B_1 + (\beta - B_3)A^2]^{1/2}, \quad (21a)$$

$$k = \{(\beta - B_3)A^2/2[1 - B_1 + (\beta - B_3)A^2]\}^{1/2}. \quad (21b)$$

The favored forcing function is

$$F(\tau) = B_1A \operatorname{cn}(p\tau, k) + B_3A^3 \operatorname{cn}^3(p\tau, k). \quad (22)$$

Here we note that five parameters are involved in the solution, namely: A, B_1, B_3, p , and k , and there are two relationships between them. Therefore any three of the five, except the group of B_1, p , and k , can be arbitrarily chosen. A simple elliptic response like (21) is assured when the other two parameters are determined from (21a) and (21b). For the convenience of future applications we shall write out the favored elliptic forcing function, assuming that p, k and A have been chosen arbitrarily. This results in

$$F(\tau) = A[1 - p^2(1 - 2k^2)] \operatorname{cn}(p\tau, k) + A(\beta A^2 - 2p^2k^2) \operatorname{cn}^3(p\tau, k). \quad (23)$$

Another way of looking at the solution is the following: Let us denote B_1A by C_1 and B_3A^3 by C_3 . They are respectively the coefficients of the linear and cubic terms in (23). Combining C_1 and C_3 by eliminating A we obtain

$$\beta^{1/2}C_3 = \frac{\beta^{1/2}C_1}{1 - p^2(1 - 2k^2)} \left\{ \frac{\beta C_1^2}{[1 - p^2(1 - 2k^2)]^2} - 2p^2k^2 \right\} \quad (24)$$

as the necessary relationship between C_1, C_3, p , and k of the forcing function in order to have a simple elliptic response.

It should be pointed out that although the exact solutions discussed here require a special class of forcing functions, that class itself is fairly general because of the freedom to choose any three of the parameters C_1, C_3, p , and k at will.

3.3 Multiple-term elliptic response. The inverse method employed to find exact solutions in the last two sections can be extended to those consisting of more than one elliptic term. For example, if we require the solution to be

$$x = A_1 \operatorname{cn}(p\tau, k) + A_3 \operatorname{cn}^3(p\tau, k), \quad (25)$$

the necessary forcing function can be found by substituting (25) into the left side of (11). Before doing this it is, however, convenient to perform a transformation on (11). Putting $y = cn(p\tau, k)$ and $X(y) = x(\tau)$, Eq. (11) becomes

$$p^2[(1 - k^2) - (1 - 2k^2)y^2 - k^2y^4] \frac{d^2X}{dy^2} + p^2[-(1 - 2k^2)y - 2k^2y^3] \frac{dX}{dy} + X + \beta X^3 = G(y), \quad (26)$$

where $G(y) = F(\tau)$.

In our case (25), $X = A_1y + A_3y^3$. When this is substituted in (26) and all terms are grouped in powers of y we get

$$G(y) = D_1y + D_3y^3 + D_5y^5 + D_7y^7 + D_9y^9, \quad (27)$$

where

$$\begin{aligned} D_1 &= [1 - p^2(1 - 2k^2)]A_1 + 6p^2(1 - k^2)A_3, \\ D_3 &= [\beta A_1^2 - 2p^2k^2]A_1 + [1 - 9p^2(1 - 2k^2)]A_3, \\ D_5 &= 3[\beta A_1^2 - 4p^2k^2]A_3, \\ D_7 &= 3\beta A_1A_3^2, \\ D_9 &= \beta A_3^3. \end{aligned}$$

Again, in this solution four parameters out of a total of nine may be arbitrarily chosen. An application of this exact solution will be discussed in the next section. It is evident how this result can easily be extended to cases where $X(y)$ is of power higher than three. In general, if $X(y)$ is of power $2n - 1$ there will be $4n + 1$ parameters and $3n - 1$ relations between them. Hence $n + 2$ parameters may be chosen arbitrarily.

4. Simple harmonic forcing. Up to now we have only discussed some exact solutions under external forcing consisting of favored elliptic functions. When the forcing function is simple harmonic the equation of motion is

$$\frac{d^2x}{d\tau^2} + x + \beta x^3 = P_0 \cos \frac{2\pi}{T} \tau = P_0 \cos \omega \tau, \quad (28)$$

where T is the forcing period, and ω the forcing frequency. In this instance an exact solution in a simple form is hardly to be expected. Various procedures are available to find the approximate steady state solution.

One method consists in assuming as an approximate solution a suitable function $z(\tau)$ with some undetermined parameters, and then defining an error function $e(\tau)$ as,

$$e(\tau) = \frac{d^2z}{d\tau^2} + z + \beta z^3 - P_0 \cos \frac{2\pi}{T} \tau \quad (29)$$

and, finally, determining the parameters such that the total error specified in some fashion be a minimum. From what we have learned in the preceding sections it seems natural to take a suitable elliptic function as the approximate solution. Let us put $z(\tau) = A \operatorname{cn}(p\tau, k)$. The error function is then given by

$$e(\tau) = AB \operatorname{cn}(p\tau, k) - P_0 \cos \frac{2\pi}{T} \tau \quad (30)$$

provided that Eqs. (14) are satisfied. Furthermore, to assure the same period for the response and the forcing function we must have

$$p = 4K(k)/T. \quad (31)$$

Here we have four parameters A , B , p , and k , but there are three conditions, Eqs. (14) and Eq. (31), to be satisfied. Only one of them is completely at our disposal. This one can be determined by using any of the several minimum total error criteria, such as the collocation method, the Galerkin's method, or the least square method. Solutions obtainable in this fashion were computed. When k^2 is small, which means small $|\beta A^2|$, these methods yield the well-known result

$$\omega^2 = 1 + \frac{3}{4}\beta A^2 - \frac{P_0}{A}.$$

When a multiple-term elliptic response is used for the approximate solution $z(\tau)$, proper formulation of the minimum total error can be made in the usual manner without difficulty, although the actual calculation becomes very involved, as usual.

We shall not reproduce these solutions here. Instead we wish to derive the approximate solutions of various orders of the equation (28) entirely from the various exact solutions obtained in Sec. 3. The central idea is to compare the orders of magnitude of the terms appearing in the exact solutions. It will be shown that approximate solutions of different orders will result quite naturally from this kind of analysis.

4.1 First order approximation. From Sec. 3.1 it is seen that the response will be $x = A \operatorname{cn}(p\tau, k)$, provided that the forcing function is of the form of (16). This favored elliptic forcing function will be rewritten here as $F_1(\tau)$.

$$F_1(\tau) = A[1 - p^2(1 - 2k^2)] \operatorname{cn}(p\tau, k). \quad (32)$$

The function cn may be expanded into the Fourier series

$$\operatorname{cn}(p\tau, k) = \sum_{n=0}^{\infty} a_{2n+1} \cos \frac{(2n+1)\pi p\tau}{2K}, \quad a_{2n+1} = \frac{2\pi}{kK} \frac{q^{n+1/2}}{1 + q^{2n+1}}. \quad (33)$$

In this expansion it may be easily verified that the magnitudes of the various components are of the orders: $1, k^2, k^4, \dots$ when k^2 is small in comparison with unity. Hence, outside of a common factor A the orders of magnitude of the leading parts of various harmonic components in $F_1(\tau)$ are respectively: $(1 - p^2)$, $(1 - p^2)k^2$, $(1 - p^2)k^4$, \dots . From (14), βA^2 is known to be of the order k^2 .

Next, let us see what error we commit if we replace the simple harmonic forcing function $P_0 \cos \omega\tau$ by $F_1(\tau)$. When $1 - p^2$ is restricted to be of the order of k^2 , the first harmonic in $F_1(\tau)$ will be of the order k^2 and the rest will all be of higher order. By setting the coefficient of the first harmonic equal to P_0 , we find

$$A[1 - p^2(1 - 2k^2)] = P_0. \quad (34)$$

We insure an accuracy of k^2 if we replace $P_0 \cos \omega\tau$ by $F_1(\tau)$, provided of course that k^2 is small compared with unity. In order to have the same periodicity for the response and the forcing we must have

$$p = 2\omega K/\pi. \quad (35)$$

Equations (34), (35) and the first equation of (14) completely determine the values of

A , p , and k , thus the first order approximate solution, for any set of assigned values of β , ω , P_0 . Eliminating p from these three equations and expanding K in powers of k^2 , we get

$$\beta A^2 - 2k^2\omega^2[1 + \frac{1}{2}k^2 + O(k^4)] = 0, \quad (36)$$

$$(1 - \omega^2) - \frac{1}{16}k^2(1 - 25\omega^2) + O(k^4) = \frac{P_0}{A}. \quad (37)$$

Since, in this analysis, we limit $1 - p^2$ to the order of k^2 , we find that $1 - \omega^2$ is also of the order k^2 and may be taken as αk^2 in a comparison of orders of magnitude, α being a quantity of the order unity. As the forcing function is approximated to the accuracy of k^2 , only terms up to that order will be retained in (36) and (37). These equations now become

$$\beta A^2 - 2k^2 = 0, \quad (36a)$$

$$\alpha k^2 + \frac{3}{2}k^2 = \frac{P_0}{A}. \quad (37a)$$

From them we find

$$\omega^2 = 1 + \frac{3}{4}\beta A^2 - \frac{P_0}{A}, \quad \text{or,} \quad \omega = 1 + \frac{1}{2}\left[\frac{3}{4}\beta A^2 - \frac{P_0}{A}\right]. \quad (38)$$

The response is $x = A \operatorname{cn}(p\tau, k)$. Again, retaining only terms up to k^2 in its Fourier expansion, one gets

$$x = A\left\{\cos \omega\tau + \frac{1}{32}\beta A^2(-\cos \omega\tau + \cos 3\omega\tau)\right\}. \quad (39)$$

Equations (38) and (39) are exactly those obtained from the perturbation method, [8]. Although this first order solution result is not new, the method through which it is obtained here is novel in that it is obtained as a special case from an exact solution of forced oscillation. Throughout the derivation one thing stands out. In the perturbation method the perturbation parameter usually used is the coefficient β of the non-linear term. The present analysis indicates that the controlling parameter is k^2 or $\beta A^2/2$ rather than β . Physically, this is to be expected. Even when β is not small, but A is restricted to be small, the first order solution is still valid.

Because of the requirement that $1 - \omega^2$ be small, the solution obtained is only valid in the neighborhood of linear free vibrations. Let us now drop this requirement and assume that $1 - p^2$ may be of the order unity. In this case it is necessary to start with the exact solution found in Sec. 3.2. The response is again $x = A \operatorname{cn}(p\tau, k)$ and the favored elliptic forcing function is given by (23) and will be denoted here as $F_2(\tau)$.

The function cn^3 may be expanded into a Fourier series.

$$\operatorname{cn}^3(p\tau, k) = \sum_{n=0}^{\infty} b_{2n+1} \cos \frac{(2n+1)\pi p\tau}{2K}, \quad (40)$$

$$b_{2n+1} = \frac{1}{2k^2} \left\{ 2k^2 - 1 + (2n+1)^2 \left(\frac{\pi}{2K} \right)^2 \right\} \frac{2\pi}{kK} \frac{q^{n+1/2}}{1 + q^{2n+1}}.$$

When k^2 is small the magnitudes of the various components are: 1, 1, k^2 , k^4 , \dots . It will

be shown later *a posteriori* that when k^2 is small, βA^2 is of the order k^2 . Keeping this in mind we find that, apart from a common factor A , the orders of magnitude of various harmonic components in $F_2(\tau)$ are:

$$\begin{array}{ccccccc} (1-p^2), & (1-p^2)k^2, & (1-p^2)k^4, & \dots & \text{from } cn \text{ term.} \\ k^2, & k^2, & k^4, & \dots & \text{from } cn^3 \text{ term,} \end{array}$$

If we again neglect terms of order k^4 or higher we have two harmonics from the cn and two from the cn^3 . By reasoning similar to that used previously we can approximate the forcing function $P_0 \cos \omega \tau$ by $F_2(\tau)$ with an accuracy up to k^2 , if we set

$$\begin{aligned} [1 - p^2(1 - 2k^2)]a_1 + (\beta A^2 - 2k^2 p^2)b_1 &= P_0/A, \\ [1 - p^2(1 - 2k^2)]a_3 + (\beta A^2 - 2k^2 p^2)b_3 &= 0. \end{aligned} \quad (41)$$

Another necessary condition is that concerning the periodicity given by (35). Equations (41) and (35) completely determine the approximate solution, provided that k thus found satisfies the order-of-magnitude requirement.

Substituting the coefficients a_1 , a_3 , b_1 , and b_3 from (33) and (40) into (41), eliminating p by using (35), expanding K into powers of k^2 , and retaining only terms up to k^2 , we get

$$\begin{aligned} (1 - \omega^2) - (1 - 25\omega^2)k^2/16 + 3(\beta A^2 - 2k^2\omega^2)/4 &= P_0/A, \\ (1 - \omega^2)k^2/4 + (\beta A^2 - 2k^2\omega^2) &= 0. \end{aligned} \quad (42)$$

Eliminating k^2 from these two, we have

$$1 - \omega^2 - \frac{P_0}{A} + \beta A^2 \left(\frac{7\omega^2 - 1}{9\omega^2 - 1} \right) = 0. \quad (43)$$

After A has been determined from (43) for given β , ω , and P_0 , k^2 may be found from (42).

$$k^2 = 4\beta A^2 / (9\omega^2 - 1). \quad (44)$$

The response is given by the following expression.

$$x = A \left\{ \cos \omega \tau + \frac{k^2}{16} (-\cos \omega \tau + \cos 3\omega \tau) \right\}. \quad (45)$$

Near $\omega = 1/3$ the solution is probably not valid because of the large value of k^2 which invalidates the analysis. This solution also requires that $|\beta A^2|$ be small in comparison with unity. It follows from (43) that $1 - \omega^2 - P_0/A$ has to be small, implying small deviations from the linear forced response. The solution obviously represents a perturbation from the linear forced vibration. It is interesting to note that in most of the frequency ranges (43) does not differ very much from (38).

4.2 Higher order approximations. In the exact solution under two-term favored elliptic forcing we have three parameters completely at our disposal. However, in approximating a simple harmonic forcing by such an elliptic forcing function, one relation between these three is necessary on account of the periodicity requirement. With only two parameters at our disposal we can not hope to obtain any approximation beyond the zeroth order and the order of k^2 . In order to get approximations of higher order we resort to the exact solutions of multiple-term elliptic response discussed in Sec. 3.3. For example, for the second order approximation we may use the response given by (25). The necessary forcing function is rewritten here as $F_3(\tau)$.

$$F_3(\tau) = D_1 \operatorname{cn} + D_3 \operatorname{cn}^3 + D_5 \operatorname{cn}^5 + D_7 \operatorname{cn}^7 + D_9 \operatorname{cn}^9, \quad (46)$$

where cn stands for $\operatorname{cn}(p\tau, k)$ and the D 's are given by (27).

Let us assume that βA^2 is of the order k^2 and A_3 is small in comparison with A_1 so that (A_3/A_1) is also of the order k^2 . We shall denote (A_3/A_1) by $\alpha_3 k^2$ in order-of-magnitude calculations, α_3 being a quantity of order unity. Under these circumstances the coefficients D_7 and D_9 will be of the orders $k^6 A$ and $k^8 A$ respectively. If we retain only terms with powers of up to k^4 , the D_7 and D_9 terms need not be considered. The orders of D_1 , D_3 , and D_5 are respectively A , $k^2 A$ and $k^4 A$.

The function cn^5 may be expanded into a Fourier series. Let us denote the Fourier coefficients by c_{2n+1} . They are respectively of the order 1, 1, 1, k^2 , \dots when k^2 is small. Again by comparing the orders of magnitude of various terms in $F_3(\tau)$ we can approximate the given forcing function $P_0 \cos \omega\tau$ by $F_3(\tau)$ to the accuracy of k^4 , provided that we have

$$\begin{aligned} D_1 a_1 + D_3 b_1 + D_5 c_1 &= P_0, \\ D_1 a_3 + D_3 b_3 + D_5 c_3 &= 0, \\ D_1 a_5 + D_3 b_5 + D_5 c_5 &= 0. \end{aligned} \quad (47)$$

Retaining only the necessary powers of k in a 's, b 's and c 's to maintain a uniform accuracy of k^4 in (47), we have

$$\begin{aligned} a_1 &= 1 - \frac{1}{16} k^2 - \frac{9}{256} k^4 + O(k^6), \\ a_3 &= \frac{1}{16} k^2 + \frac{1}{32} k^4 + O(k^6), \\ a_5 &= \frac{1}{256} k^4 + O(k^6), \\ b_1 &= \frac{3}{4} - \frac{3}{32} k^2 + O(k^4), \\ b_3 &= \frac{1}{4} + \frac{3}{64} k^2 + O(k^4), \\ b_5 &= \frac{3}{64} k^2 + O(k^4), \\ c_1 &= \frac{10}{16} + O(k^2), \\ c_3 &= \frac{5}{16} + O(k^2), \\ c_5 &= \frac{1}{16} + O(k^2). \end{aligned} \quad (48)$$

Substituting these coefficients and the expressions for D 's from (27) into (47) and retaining only terms up to k^4 , we get

$$\begin{aligned}
 1 - \omega^2 + \frac{3}{4}\beta A_1^2 + \frac{k^2}{16}(1 - \omega^2)(12\alpha_3 - 1) + \frac{3}{32}k^2\beta A_1^2(20\alpha_3 - 1) \\
 - \frac{3}{256}k^4(1 - \omega^2)(8\alpha_3 + 3) = P_0/A_1, \\
 \beta A_1^2 + \frac{1}{4}k^2(1 - 9\omega^2)(1 + 4\alpha_3) + \frac{15}{4}\alpha_3 k^2\beta A_1^2 + \frac{1}{16}k^4(1 - 9\omega^2)(2 + 3\alpha_3) = 0, \\
 k^2\beta A_1^2(1 + 4\alpha_3) + \frac{1}{12}k^4(1 - 25\omega^2)(1 + 12\alpha_3) = 0.
 \end{aligned} \tag{49}$$

Here the periodicity requirement (35) has been used to eliminate p^2 in the D 's. These three equations completely determine the second order approximation. In principle at least, the values of A_1 , α_3 , and k^2 can be determined for any set of assigned values of P_0 , ω , and β . The solution is valid only when the resulting k^2 is small and α_3 is of the order unity. Once the values of A_1 , α_3 and k^2 are known, the response is given with an accuracy up to k^4 by

$$\begin{aligned}
 x = A_1 \left\{ \left[1 + k^2 \left(-\frac{1}{16} + \frac{3}{4}\alpha_3 \right) + k^4 \left(-\frac{9}{256} - \frac{3}{32}\alpha_3 \right) \right] \cos \omega\tau + \left[k^2 \left(\frac{1}{16} + \frac{1}{4}\alpha_3 \right) \right. \right. \\
 \left. \left. + k^4 \left(\frac{1}{32} + \frac{3}{64}\alpha_3 \right) \right] \cos 3\omega\tau + \left[k^4 \left(\frac{1}{256} + \frac{3}{64}\alpha_3 \right) \right] \cos 5\omega\tau + O(k^6) \right\}.
 \end{aligned} \tag{50}$$

For approximations of higher order than the second, similar analyses may be made by employing solutions with three or more terms in the response. No essential difficulties will arise except the complexity of the mathematical computation.

Before concluding this section we summarize the essential features of the analysis presented here. Our aim is to find a steady state solution of (28). The general approach given here is, first to find some exact solutions of slightly different problems; in the present problem, namely: elliptic responses under favored elliptic forcings. At the first sight one may think that these exact solutions are very restricted. In actuality they represent a very wide class of solutions because of the freedom to assign arbitrary values to the parameters available at our disposal. Indeed, approximate solutions of various orders of (28) are but special cases of these exact solutions when k^2 is small. Furthermore they can be obtained simply by comparing the orders of magnitude of various terms in the expansions without having to solve new differential equations which appear in the perturbation method.

It is believed that this type of approach may find application in solving other non-linear problems.

5. Subharmonic solutions and simple elliptic subharmonics. Consider again the simple elliptic response solution under two-term elliptic forcing, discussed in Sec. 3.2. The functions cn and cn^3 may be expanded into Fourier series according to (33) and (40). The lowest components are respectively:

$$\begin{aligned}
 B_1 A \frac{2\pi}{kK} \frac{q^{1/2}}{1+q} \cos \frac{\pi p\tau}{2K} \\
 B_3 A^3 \frac{1}{2k^3} \left\{ 2k^2 - 1 + \left(\frac{\pi}{2K} \right)^2 \right\} \frac{2\pi}{kK} \frac{q^{1/2}}{1+q} \cos \frac{\pi p\tau}{2K}.
 \end{aligned}$$

If we set these two equal in magnitude and opposite in sign, the fundamental component vanishes and there results

$$\frac{B_1}{B_3 A^2} = - \left\{ \frac{2k^2 - 1}{2k^2} + \frac{1}{2k^2} \left(\frac{\pi}{2K} \right)^2 \right\}. \quad (51)$$

Equation (51) expressed in terms of p , k , and A implies

$$\beta A^2 = 2k^2 p^2 - 2k^2 [1 - p^2(1 - 2k^2)] / \left[2k^2 - 1 + \left(\frac{\pi}{2K} \right)^2 \right]. \quad (52)$$

Under this condition the forcing function is devoid of the fundamental component and yet the same component in the response is predominant. This is the well-known *subharmonic phenomenon*.

When $k = 0$, we have

$$\begin{aligned} B_1 &= 1 - p^2, & B_3 &= -\beta A^2 = \frac{4}{3}(1 - p^2), \\ x &= A \cos p\tau, & F(\tau) &= -\frac{1}{3}A(1 - p^2) \cos 3p\tau, \end{aligned} \quad (53)$$

which is the case of simple subharmonic of order $1/3$.

Similar analyses can be made for solutions with the sn function. It is to be noted here that although (52) leads to subharmonic solutions which are neither "pure" nor "simple" as defined by Rosenberg [6], they are nevertheless *exact* solutions.

5.1 Simple elliptic subharmonic solutions. Analogous to the simple subharmonics we can define *simple elliptic subharmonic* solution of order $1/r$ as that $x = A \operatorname{cn}(p\tau, k)$ excited by a forcing function $F(\tau) = C \operatorname{cn}(r p \tau, k)$. The condition under which such solutions exist can again be obtained by either the inverse method [7] or the technique of transformation described in [4].

Let us call $\operatorname{cn}(v, k)$ as z and define a function $H_n(z)$ according to

$$H_n(z) = \operatorname{cn}(nv, k). \quad (54)$$

This is a generalization of the Tehebycheff's polynomials of the first kind. The general function $H_n(z)$ may be found by the addition theorems of elliptic functions.

Next let us consider a solution $\eta = \operatorname{cn}(u/r, k)$. It satisfies the following differential equation

$$\frac{d^2 \eta}{du^2} + \frac{1}{r^2} (1 - 2k^2) \eta + \frac{2k^2}{r^2} \eta^3 = 0. \quad (55)$$

If we add to this a trivial identity

$$gH_r(\eta) = g \operatorname{cn}(u, k) \quad (56)$$

we get a differential equation

$$\frac{d^2 \eta}{du^2} + \frac{1}{r^2} (1 - 2k^2) \eta + \frac{2k^2}{r^2} \eta^3 + gH_r(\eta) = g \operatorname{cn}(u, k) \quad (57)$$

which represents a special non-linear forced oscillation under an elliptic forcing. The response $\operatorname{cn}(u/r, k)$ is also a simple elliptic function but of a period r times as large as the forcing function. This is defined as a simple elliptic subharmonic of order $1/r$.

In general $H_n(z)$ will not be a finite polynomial but rather a more general rational

function in z . For example, $H_3(z)$ is given by

$$H_3(z) = \frac{z[1 - 4s^2 + k^2(6s^4) + k^4(-4s^6 + s^8)]}{1 + k^2(4s^6 - 6s^4) + k^4(4s^8 - 3s^6)}, \quad (58)$$

where $s^2 = sn^2(v, k) = 1 - z^2$. When the modulus k^2 is small, $H_n(z)$ can however be expanded into a series in powers of k^2 . Thus, for $n = 3$ we have

$$H_3(z) = z[(-3 + 4z^2) + 4k^2z^2(-1 + 6z^2 - 9z^4 + 4z^6) + 4k^4z^2(1 - z^2)^3(-1 + 3z^2 + 12z^4 - 16z^6) + O(k^6)]. \quad (59)$$

Substituting this into (57) with $r = 3$ and retaining only terms up to k^2 , we get

$$\begin{aligned} \frac{d^2\eta}{du^2} + \left(\frac{1 - 2k^2}{9} - 3g\right)\eta + \left[\frac{2k^2}{9} + 4g(1 - k^2)\right]\eta^3 + 24k^2g\eta^5 \\ - 36k^2g\eta^7 + 16k^2g\eta^9 = g \operatorname{cn}(u, k), \end{aligned} \quad (60)$$

as the differential equation which possesses a simple elliptic subharmonic solution of the order $1/3$. The solution is of the accuracy k^2 , where k^2 is assumed to be small.

When $k = 0$, the function $H_n(z)$ reduces to the usual Tchebycheff's polynomials of the first kind and Eq. (57) becomes the one given by Rosenberg in [6].

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BOOK REVIEWS

(Continued from p. 374)

solved by the intelligent application of simple arithmetic". This policy has led to a choice of problems which require only a direct application of the discussed methods and formulas, and which therefore serve as useful exercises in probability algebra. Unfortunately, there is a corresponding artificiality to the problems, and there are very few which are either of a really imaginative nature or which lead to thought concerning their deduction from a real situation. On the other hand, this feature would not detract too much from the use of the book as a companion text in a "case-work" kind of course.

The first chapter provides a survey of probability concepts; although the authors do not hesitate to discuss "sample spaces" or "interaction and union of derived events", they rather paradoxically omit the derivation of the Poisson distribution formula to be used throughout the book. Also, the occasional dark comments as to the inherent mathematical difficulty of probability do not seem pedagogically useful. In the chapter on Allocation, the simplex method is presented rather clearly, but with some mathematical awkwardness (cf. the incorrect statement of the key theorem of linear programming on p. 226.) As a final minor objection, the point of the "principle of optimality" (p. 272) in the chapter on dynamic programming is quite lost on the reviewer.

CARL E. PEARSON

Instationäre Wärmespannungen. By Heinz Parkus. Springer-Verlag, Vienna, 1959. v + 165 pp. \$9.05.

This book is a sequel to a previous volume on steady-state thermal stresses by E. Melan and the present author. It supplements the earlier treatise by providing a fairly extensive and up-to-date account of available investigations of thermal stresses and deformations due to time-dependent temperature fields. While most of the material selected here belongs to classical elasticity theory, the book ends with an excursion into the fields of viscoelastic and elasto-plastic solids.

The opening chapter contains certain elements of the linear theory of thermoelasticity for homogeneous, isotropic media. As in the preceding volume, however, the primary emphasis is not so much on a systematic and complete exposition of basic theory as on a careful and detailed discussion of solutions to particular problems. Chapter II deals with investigations of transient temperature stresses pertaining primarily to an unbounded domain, the half-space, the sphere, and the circular cylinder. Chapter III is devoted to the effect of periodic temperature changes, whereas problems involving moving heat sources are treated in Chapter IV. These quasi-static analyses are followed by a chapter on inertia effects in transient thermoelasticity which presents a resumé of recent studies in this area. Thermal stresses in viscoelastic and elasto-plastic media are taken up in two concluding chapters.

Some of the results included by the author are in the form of improper integrals. It is difficult to appreciate the usefulness and significance of such solutions to boundary-value problems in the absence of adequate numerical evaluations which are missing in several important instances. On the other hand, integral representations of singular solutions which correspond to thermoelastic nuclei are bound to be of limited interest in the further development of thermoelasticity theory.

A brief discussion of thermoelastic coupling effects might well have been appropriate in an enterprise of this kind. Further, the chapters on temperature stresses in elastic materials are rather sketchy and reflect the ritualistic character of our present approach to such problems—an approach that is for the most part safely removed from the complexities of the actual physical situation at which it aims. A more realistic treatment of thermo-inelasticity cannot escape the essential nonlinearities attached to this topic and must come to grips with the temperature-dependence of the physical parameters which govern the thermal and mechanical behavior of such materials at elevated temperatures.

It is hardly fair, however, to blame the author, who has made a very welcome contribution to a very timely subject, for inadequacies in the present state of this subject. The book is clearly written, carefully edited, and admirably printed. The comprehensive bibliography appearing at the end of the volume will be greatly appreciated by every interested reader.

ELI STERNBERG

TORSION AND EXTENSION OF HELICOIDAL SHELLS*

JAMES K. KNOWLES (*California Institute of Technology*)

AND

ERIC REISSNER (*Massachusetts Institute of Technology*)

1. Introduction. The present paper is concerned with the problem of rotationally symmetric deformations of thin elastic shells the middle surface of which is a portion of a right helicoid. Particular consideration is given to the problem of a uniformly pretwisted thin strip which is acted upon by tractions which result in equal and opposite axial forces F , and equal and opposite axial torques T (Fig. 1).

The differential equations for stresses and deformations of thin homogeneous, isotropic helicoidal shells, as used here, have been derived elsewhere [1]. In what follows they are employed in the form which they assume for rotationally symmetric states of stress and strain. Insofar as the problem of the pretwisted strip is concerned one of the essential aspects of the analysis is the connection between rotationally symmetric states of strain which depend on states of displacement which are not rotationally symmetric. The details of this connection are established in the present paper.

Of particular interest in the problem of the pretwisted strip are the relations between the applied force and torque on the one hand and the angle of elastic twist and the relative axial extension on the other hand. In this connection certain explicit results are presented which generalize earlier work of Chen Chu [2] regarding the torsional rigidity of the strip.

2. Equations for helicoidal shells. Let r, θ, z be cylindrical coordinates and let

$$z = a\theta \quad (2.1)$$

be the equation of the middle surface of the shell. The constant $2\pi a$ is the pitch of the helicoidal middle surface. The parametric curves $r = \text{constant}$ and $\theta = \text{constant}$ on the middle surface of the shell form an orthogonal net but are not the lines of curvature.

The state of stress in the helicoidal shell (2.1) is described by stress resultants $N_r, N_\theta, N_{r\theta}, N_{\theta r}, Q_r$ and Q_θ and stress couples $M_r, M_\theta, M_{r\theta}$ and $M_{\theta r}$, referred to tangential and normal directions at the edges of an element of the shell (Fig. 2). The differential equations of equilibrium of an element of the shell are [1],

$$\frac{\partial}{\partial r}(\alpha N_r) + \frac{\partial N_{\theta r}}{\partial \theta} - \frac{r}{\alpha} N_\theta + \frac{a}{\alpha} Q_\theta = 0, \quad (2.2)$$

$$\frac{\partial}{\partial r}(\alpha N_{r\theta}) + \frac{\partial N_\theta}{\partial \theta} + \frac{r}{\alpha} N_{\theta r} + \frac{a}{\alpha} Q_r = 0, \quad (2.3)$$

$$\frac{\partial}{\partial r}(\alpha Q_r) + \frac{\partial Q_\theta}{\partial \theta} - \frac{a}{\alpha} (N_{r\theta} + N_{\theta r}) = 0, \quad (2.4)$$

$$\frac{\partial}{\partial r}(\alpha M_r) + \frac{\partial M_{\theta r}}{\partial \theta} - \frac{r}{\alpha} M_\theta - \alpha Q_r = 0, \quad (2.5)$$

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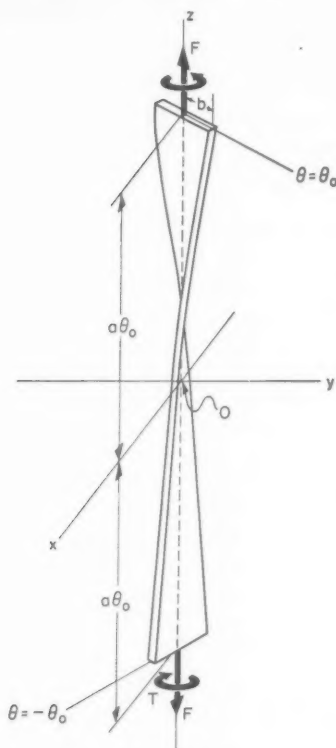


FIG. 1 Pretwisted strip

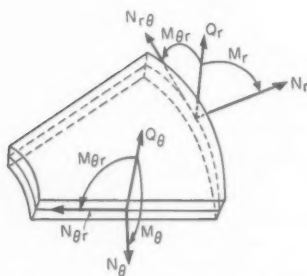


FIG. 2 Element of helicoidal shell

$$\frac{\partial}{\partial r} (\alpha M_{r\theta}) + \frac{\partial M_\theta}{\partial \theta} + \frac{r}{\alpha} M_{\theta r} - \alpha Q_\theta = 0, \quad (2.6)$$

$$N_{r\theta} - N_{\theta r} + \frac{\alpha}{r} (M_\theta - M_r) = 0. \quad (2.7)$$

The quantity α is defined by

$$\alpha = (a^2 + r^2)^{1/2}. \quad (2.8)$$

The system (2.2) to (2.7) is completed by a system of stress strain relations which here is taken in a form corresponding to the relations of Flügge [3] and Byrne [4] for lines of curvature coordinates and which is [1],

$$N_r = \frac{Eh}{1-\nu^2} (\epsilon_r + \nu\epsilon_\theta) - \frac{a}{\alpha^2} \frac{Eh^3}{12(1-\nu^2)} \left[\tau^* - (1-\nu) \frac{a}{\alpha^2} (\epsilon_r - \epsilon_\theta) \right], \quad (2.9)$$

$$N_\theta = \frac{Eh}{1-\nu^2} (\epsilon_\theta + \nu\epsilon_r) - \frac{a}{\alpha^2} \frac{Eh^3}{12(1-\nu^2)} \left[\tau^* - (1-\nu) \frac{a}{\alpha^2} (\epsilon_\theta - \epsilon_r) \right], \quad (2.10)$$

$$N_{r\theta} = \frac{Eh}{2(1+\nu)} \gamma_{r\theta} - \frac{a}{\alpha^2} \frac{Eh^3}{24(1-\nu^2)} [(1+\nu)\kappa_r^* + (3-\nu)\kappa_\theta^*], \quad (2.11)$$

$$N_{\theta r} = \frac{Eh}{2(1+\nu)} \gamma_{r\theta} - \frac{a}{\alpha^2} \frac{Eh^3}{24(1-\nu^2)} [(1+\nu)\kappa_\theta^* + (3-\nu)\kappa_r^*], \quad (2.12)$$

$$M_r = \frac{Eh^3}{12(1-\nu^2)} \left(\kappa_r^* + \nu\kappa_\theta^* - \frac{1-\nu}{2} \frac{a}{\alpha^2} \gamma_{r\theta} \right), \quad (2.13)$$

$$M_\theta = \frac{Eh^3}{12(1-\nu^2)} \left(\kappa_\theta^* + \nu\kappa_r^* - \frac{1-\nu}{2} \frac{a}{\alpha^2} \gamma_{r\theta} \right), \quad (2.14)$$

$$M_{r\theta} = \frac{Eh^3}{12(1-\nu^2)} \left[\frac{1-\nu}{2} \tau^* - \frac{a}{\alpha^2} (\epsilon_\theta + \nu\epsilon_r) \right], \quad (2.15)$$

$$M_{\theta r} = \frac{Eh^3}{12(1-\nu^2)} \left[\frac{1-\nu}{2} \tau^* - \frac{a}{\alpha^2} (\epsilon_r + \nu\epsilon_\theta) \right]. \quad (2.16)$$

The strain quantities ϵ , γ , κ and τ are expressed in terms of *radial*, *circumferential* and *axial* components of middle surface displacement u , v and w as follows;

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad (2.17)$$

$$\epsilon_\theta = \frac{r}{\alpha^2} \frac{\partial v}{\partial \theta} + \frac{a}{\alpha^2} \frac{\partial w}{\partial \theta} + \frac{r}{\alpha^2} u, \quad (2.18)$$

$$\gamma_{r\theta} = \frac{1}{\alpha} \frac{\partial u}{\partial \theta} + \frac{r}{\alpha} \frac{\partial v}{\partial r} + \frac{a}{\alpha} \frac{\partial w}{\partial r} - \frac{v}{\alpha}, \quad (2.19)$$

$$\kappa_r^* = -\frac{r}{\alpha} \frac{\partial^2 w}{\partial r^2} - \frac{a^2}{\alpha^3} \frac{\partial w}{\partial r} + \frac{a}{\alpha} \frac{\partial^2 v}{\partial r^2} - \frac{ar}{\alpha^3} \frac{\partial v}{\partial r} + \frac{a}{\alpha^3} v - \frac{a}{\alpha^3} \frac{\partial u}{\partial \theta}, \quad (2.20)$$

$$\kappa_\theta^* = -\frac{r}{\alpha^3} \frac{\partial^2 w}{\partial \theta^2} + \frac{a}{\alpha^3} \frac{\partial^2 v}{\partial \theta^2} - \frac{1}{\alpha} \frac{\partial w}{\partial r} + \frac{a}{\alpha^3} \frac{\partial u}{\partial \theta}, \quad (2.21)$$

$$\tau^* = -\frac{2r}{\alpha^3} \frac{\partial^2 w}{\partial r \partial \theta} + \frac{2a}{\alpha^2} \frac{\partial^2 v}{\partial r \partial \theta} + \frac{2(r^2 - a^2)}{\alpha^4} \frac{\partial w}{\partial \theta} - \frac{4ar}{\alpha^4} \frac{\partial v}{\partial \theta} - \frac{4ar}{\alpha^4} u. \quad (2.22)$$

For the formulation of boundary conditions it is necessary to have expressions for *effective* edge stress resultants which include the action of the edge twisting moments and their derivatives, insofar as they are statically equivalent to distributions of edge forces. *Radial*, *circumferential* and *axial* components of these effective edge stress resultants are, along edges $r = \text{constant}$,

$$R_r = N_r + \frac{a}{\alpha^2} M_{r\theta}, \quad (2.23)$$

$$H_r = \frac{r}{\alpha} N_{r\theta} - \frac{a}{\alpha} Q_r - \frac{a}{\alpha^2} \frac{\partial M_{r\theta}}{\partial \theta}, \quad (2.24)$$

$$Z_r = \frac{a}{\alpha} N_{r\theta} + \frac{r}{\alpha} Q_r + \frac{r}{\alpha^2} \frac{\partial M_{r\theta}}{\partial \theta}, \quad (2.25)$$

and, along edges $\theta = \text{constant}$,

$$R_\theta = N_\theta + \frac{a}{\alpha^2} M_{\theta r}, \quad (2.26)$$

$$H_\theta = \frac{r}{\alpha} N_\theta - \frac{a}{\alpha} Q_\theta - \frac{\partial}{\partial r} \left(\frac{a}{\alpha} M_{\theta r} \right), \quad (2.27)$$

$$Z_\theta = \frac{a}{\alpha} N_\theta + \frac{r}{\alpha} Q_\theta + \frac{\partial}{\partial r} \left(\frac{r}{\alpha} M_{\theta r} \right). \quad (2.28)$$

In addition this procedure introduces concentrated corner forces of magnitude $\pm (M_{\theta r} + M_{r\theta})$ in the direction of the normal to the middle surface of the shell, with components $\pm (r/\alpha) (M_{r\theta} + M_{\theta r})$ and $\mp (a/\alpha) (M_{r\theta} + M_{\theta r})$ in the axial and circumferential directions, respectively.

3. Boundary conditions for the problem of the pretwisted strip. We consider a helicoidal shell with edge coordinates $r = \pm b$ and $\theta = \pm \theta_0$. The usual polar coordinate interpretation is attached to the meaning of negative values of r ; the point $(-r, \theta)$ is the image of the point (r, θ) under reflection in the z -axis.

We assume that the edges $r = \pm b$ are free of stress and that the edges $\theta = \pm \theta_0$ are acted upon by forces F and torques T in accordance with Fig. 1. We then have the following system of boundary conditions along edges $r = \text{constant}$:

$$r = \pm b: \quad R_r = H_r = M_r = Z_r = 0. \quad (3.1)$$

Along edges $\theta = \text{constant}$ the boundary conditions are taken in a form which insures that a rotationally symmetric state of stress will exist in the strip. Thus we prescribe, at $\theta = \pm \theta_0$:

$$\int_{-b}^b Z_\theta dr - \left[\frac{r}{\alpha} (M_{r\theta} + M_{\theta r}) \right]_{-b}^b = F, \quad (3.2)$$

$$\int_{-b}^b H_\theta dr + \left[\frac{a}{\alpha} (M_{r\theta} + M_{\theta r}) \right]_{-b}^b = 0, \quad (3.3)$$

$$\int_{-b}^b N_{\theta r} dr = 0, \quad (3.4)$$

$$\int_{-b}^b r H_\theta dr + \left[\frac{ar}{\alpha} (M_{r\theta} + M_{\theta r}) \right]_{-b}^b = T, \quad (3.5)$$

$$\int_{-b}^b r Z_\theta dr - \left[\frac{r^2}{\alpha} (M_{r\theta} + M_{\theta r}) \right]_{-b}^b = 0, \quad (3.6)$$

$$\int_{-b}^b M_\theta dr = 0. \quad (3.7)$$

4. Displacements and stresses for the pretwisted strip. Considerations of symmetry indicate that displacements for the pretwisted strip problem should be of the form

$u = u(r)$, $v = v_0(r)\theta$ and $w = w_0(r)\theta$. The requirement that the components of strain (2.17) to (2.22) be independent of θ leads to the conclusion that admissible displacements are of the form

$$u = u(r), \quad v = k_1 r \theta, \quad w = k_2 \theta, \quad (4.1)$$

where k_1 and k_2 are constants.

For displacements of the form (4.1) we have further that $\kappa_r^* = \kappa_\theta^* = \gamma_{r\theta} = 0$. In view of Eqs. (2.11) to (2.14) this means that the displacements (4.1) are associated with the vanishing of four resultants and couples,

$$N_r = N_{\theta r} = M_r = M_\theta = 0. \quad (4.2)$$

The equilibrium equation (2.3) then implies

$$Q_r = 0, \quad (4.3)$$

and the following system of differential equations for N_r , N_θ , $M_{r\theta}$, $M_{\theta r}$, Q_θ , and u remains

$$\alpha(\alpha N_r)' - r N_\theta + \alpha Q_\theta = 0, \quad (4.4)$$

$$\alpha(\alpha M_{r\theta})' + r M_{\theta r} - \alpha^2 Q_\theta = 0, \quad (4.5)$$

$$N_r = \frac{Eh}{1-\nu^2} (\epsilon_r + \nu \epsilon_\theta) - \frac{a}{\alpha^2} \frac{Eh^3}{12(1-\nu^2)} \left[\tau^* - (1-\nu) \frac{a}{\alpha^2} (\epsilon_r - \epsilon_\theta) \right], \quad (4.6)$$

$$N_\theta = \frac{Eh}{1-\nu^2} (\epsilon_\theta + \nu \epsilon_r) - \frac{a}{\alpha^2} \frac{Eh^3}{12(1-\nu^2)} \left[\tau^* - (1-\nu) \frac{a}{\alpha^2} (\epsilon_\theta - \epsilon_r) \right], \quad (4.7)$$

$$M_{r\theta} = \frac{Eh^3}{12(1-\nu^2)} \left[\frac{1-\nu}{2} \tau^* - \frac{a}{\alpha^2} (\epsilon_\theta + \nu \epsilon_r) \right], \quad (4.8)$$

$$M_{\theta r} = \frac{Eh^3}{12(1-\nu^2)} \left[\frac{1-\nu}{2} \tau^* - \frac{a}{\alpha^2} (\epsilon_r + \nu \epsilon_\theta) \right]. \quad (4.9)$$

In these equations primes denote differentiation with respect to r , and

$$\epsilon_r = u', \quad \epsilon_\theta = \frac{r}{\alpha^2} u + k_1 \frac{r^2}{\alpha^2} + k_2 \frac{a}{\alpha^2}, \quad (4.10)$$

$$\tau^* = -\frac{4ar}{\alpha^4} u + k_1 \frac{2a(a^2 - r^2)}{\alpha^4} + k_2 \frac{2(r^2 - a^2)}{\alpha^4}. \quad (4.11)$$

Three of the four boundary conditions (3.1) are automatically satisfied and the fourth may be written in the form

$$r = \pm b: \quad \alpha^2 N_r + \alpha M_{r\theta} = 0. \quad (4.12)$$

Of the six boundary conditions (3.2) to (3.7) four are automatically satisfied and the remaining two may be simplified (upon elimination of Q_θ) by suitable integration by parts to read

$$F = \int_{-b}^b \left(\frac{a}{\alpha} N_\theta + \frac{r^2 - a^2}{\alpha^3} M_{r\theta} + \frac{r^2}{\alpha^3} M_{\theta r} \right) dr, \quad (4.13)$$

$$T = \int_{-b}^b \left(\frac{r^2}{\alpha} N_\theta + \frac{a^3 - ar^2}{\alpha^3} M_{r\theta} + \frac{a^3}{\alpha^3} M_{\theta r} \right) dr. \quad (4.14)$$

5. Non-dimensionalization and simplification. In Eq. (4.1) we may write

$$k_1 = a\omega, \quad k_2 = a\delta, \quad (5.1)$$

where ω and δ are respectively the angle of twist and the axial extension, both per unit of axial length.

We further introduce a dimensionless displacement and dimensionless resultants and couples as follows:

$$u_0 = \frac{u}{b}, \quad (5.2)$$

$$n_r = \frac{N_r}{Eh}, \quad n_\theta = \frac{N_\theta}{Eh}, \quad q_\theta = \frac{b^2 Q_\theta}{h^2 Eh}, \quad (5.3)$$

$$m_{r\theta} = \frac{bM_{r\theta}}{Eh^3}, \quad m_{\theta r} = \frac{bM_{\theta r}}{Eh^3}. \quad (5.4)$$

The quantities u_0 , n , q , and m are considered as functions of a dimensionless coordinate ρ defined by

$$\rho = \frac{r}{b}. \quad (5.5)$$

We set finally

$$\lambda = \frac{b}{a}. \quad (5.6)$$

The parameter λ measures the pretwist of the strip and vanishes for an untwisted plate located in the xz -plane.

Introduction of (5.1) to (5.6) into Eqs. (4.4) to (4.11) transforms these equations to the following form

$$\frac{d}{d\rho} [(1 + \lambda^2 \rho^2)^{1/2} n_r] - \frac{\lambda^2 \rho}{(1 + \lambda^2 \rho^2)^{1/2}} n_\theta + \frac{1}{(1 + \lambda^2 \rho^2)^{1/2}} \frac{h^2}{b^2} q_\theta = 0, \quad (5.7)$$

$$\frac{d}{d\rho} [(1 + \lambda^2 \rho^2)^{1/2} m_{r\theta}] + \frac{\lambda^2 \rho}{(1 + \lambda^2 \rho^2)^{1/2}} m_{\theta r} - (1 + \lambda^2 \rho^2)^{1/2} q_\theta = 0, \quad (5.8)$$

$$(1 - \nu^2) n_r = \epsilon_r + \nu \epsilon_\theta - \frac{1}{12} \frac{h^2}{b^2} \frac{\lambda}{1 + \lambda^2 \rho^2} \left[b\tau^* - \frac{(1 - \nu)\lambda}{1 + \lambda^2 \rho^2} (\epsilon_r - \epsilon_\theta) \right], \quad (5.9)$$

$$(1 - \nu^2) n_\theta = \epsilon_\theta + \nu \epsilon_r - \frac{1}{12} \frac{h^2}{b^2} \frac{\lambda}{1 + \lambda^2 \rho^2} \left[b\tau^* - \frac{(1 - \nu)\lambda}{1 + \lambda^2 \rho^2} (\epsilon_\theta - \epsilon_r) \right], \quad (5.10)$$

$$12(1 - \nu^2) m_{r\theta} = \frac{1 - \nu}{2} b\tau^* - \frac{\lambda}{1 + \lambda^2 \rho^2} (\epsilon_\theta + \nu \epsilon_r), \quad (5.11)$$

$$12(1 - \nu^2) m_{\theta r} = \frac{1 - \nu}{2} b\tau^* - \frac{\lambda}{1 + \lambda^2 \rho^2} (\epsilon_r + \nu \epsilon_\theta), \quad (5.12)$$

where

$$\epsilon_r = \frac{du_0}{d\rho}, \quad \epsilon_\theta = \frac{\lambda^2 \rho}{1 + \lambda^2 \rho^2} u_0(\rho) + \frac{\lambda \rho^2 b \omega}{1 + \lambda^2 \rho^2} + \frac{\delta}{1 + \lambda^2 \rho^2}, \quad (5.13)$$

$$b\tau^* = \frac{-4\lambda^3\rho}{(1+\lambda^2\rho^2)^2}u_0 + 2b\omega \frac{1-\lambda^2\rho^2}{(1+\lambda^2\rho^2)^2} + 2\delta\lambda \frac{\lambda^2\rho^2-1}{(1+\lambda^2\rho^2)^2}. \quad (5.14)$$

The system (5.7) to (5.14) is to be solved subject to the boundary conditions (4.12), which take the dimensionless form

$$\rho = \pm 1: \quad n_r + \frac{\lambda}{1+\lambda^2\rho^2} \frac{h^2}{b^2} m_{r\theta} = 0. \quad (5.15)$$

The expressions (4.13) and (4.14) for the force F and the torque T assume the form

$$F = Ehb \int_{-1}^1 \left[\frac{n_\theta}{(1+\lambda^2\rho^2)^{1/2}} + \frac{\lambda^3\rho^2-\lambda}{(1+\lambda^2\rho^2)^{3/2}} \frac{h^2}{b^2} m_{r\theta} + \frac{\lambda^3\rho^2}{(1+\lambda^2\rho^2)^{3/2}} \frac{h^2}{b^2} m_{\theta r} \right] d\rho, \quad (5.16)$$

$$T = Ehb^2 \int_{-1}^1 \left[\frac{\lambda\rho^2}{(1+\lambda^2\rho^2)^{1/2}} n_\theta + \frac{1-\lambda^2\rho^2}{(1+\lambda^2\rho^2)^{3/2}} \frac{h^2}{b^2} m_{r\theta} + \frac{1}{(1+\lambda^2\rho^2)^{3/2}} \frac{h^2}{b^2} m_{\theta r} \right] d\rho. \quad (5.17)$$

We now assume that the pretwist parameter λ is of order of magnitude unity and not large compared with unity, and that all dimensionless resultants and couples are of the same order of magnitude. Considering the fact that $h^2/b^2 \ll 1$, we may then neglect certain of the terms in the system (5.7) to (5.17) and thus reduce it to the form

$$\frac{d}{d\rho} [(1+\lambda^2\rho^2)^{1/2} n_r] - \frac{\lambda^2\rho}{(1+\lambda^2\rho^2)^{1/2}} n_\theta = 0, \quad (5.18)$$

$$\frac{d}{d\rho} [(1+\lambda^2\rho^2)^{1/2} m_{r\theta}] + \frac{\lambda^2\rho}{(1+\lambda^2\rho^2)^{1/2}} m_{\theta r} = (1+\lambda^2\rho^2)^{1/2} q_\theta, \quad (5.19)$$

$$n_r = \frac{\epsilon_r + \nu\epsilon_\theta}{1-\nu^2}, \quad (5.20)$$

$$n_\theta = \frac{\epsilon_\theta + \nu\epsilon_r}{1-\nu^2}, \quad (5.21)$$

$$m_{r\theta} = \frac{1}{12(1-\nu^2)} \left[\frac{1-\nu}{2} b\tau^* - \frac{\lambda}{1+\lambda^2\rho^2} (\epsilon_\theta + \nu\epsilon_r) \right], \quad (5.22)$$

$$m_{\theta r} = \frac{1}{12(1-\nu^2)} \left[\frac{1-\nu}{2} b\tau^* - \frac{\lambda}{1+\lambda^2\rho^2} (\epsilon_r + \nu\epsilon_\theta) \right], \quad (5.23)$$

$$\rho = \pm 1: \quad n_r = 0, \quad (5.24)$$

$$F = Ehb \int_{-1}^1 \frac{n_\theta}{(1+\lambda^2\rho^2)^{1/2}} d\rho, \quad (5.25)$$

$$T = Ehb^2 \int_{-1}^1 \left[\frac{\lambda\rho^2}{(1+\lambda^2\rho^2)^{1/2}} n_\theta + \frac{1-\lambda^2\rho^2}{(1+\lambda^2\rho^2)^{3/2}} \frac{h^2}{b^2} m_{r\theta} + \frac{1}{(1+\lambda^2\rho^2)^{3/2}} \frac{h^2}{b^2} m_{\theta r} \right] d\rho. \quad (5.26)$$

6. Reduction of the differential equations. The resultants n_r , n_θ , and the couples $m_{r\theta}$ and $m_{\theta r}$ are expressed in terms of $u_0(\rho)$ by (5.20) to (5.23). The moment equilibrium

equation (5.19) serves to express q_θ in terms of u_0 :

$$12(1 - \nu^2)q_\theta = \frac{-\nu\lambda}{1 + \lambda^2\rho^2} \frac{d^2u_0}{d\rho^2} - \frac{(4 - 3\nu)\lambda^3\rho}{(1 + \lambda^2\rho^2)^2} \frac{du_0}{d\rho} + b\omega \frac{(1 - \nu)\lambda^4\rho^3 - 2(3 - \nu)\lambda^2\rho}{(1 + \lambda^2\rho^2)^3} + \delta \frac{(7 - 5\nu)\lambda^3\rho}{(1 + \lambda^2\rho^2)^3}. \quad (6.1)$$

The remaining condition (5.18) for radial force equilibrium provides the following differential equation for u_0 .

$$\frac{d^2u_0}{d\rho^2} + \frac{\lambda^2\rho}{1 + \lambda^2\rho^2} \frac{du_0}{d\rho} + \frac{\nu\lambda^2 - \lambda^4\rho^2}{(1 + \lambda^2\rho^2)^2} u_0 = b\omega \left[\frac{(1 - \nu)\lambda\rho}{1 + \lambda^2\rho^2} - \frac{(1 + \nu)\lambda\rho}{(1 + \lambda^2\rho^2)^2} \right] + \delta \frac{(1 + \nu)\lambda^2\rho}{(1 + \lambda^2\rho^2)^2}. \quad (6.2)$$

The boundary condition (5.24) then becomes

$$\rho = \pm 1: \quad \frac{du_0}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2} u_0 + b\omega \frac{\nu\lambda\rho^2}{1 + \lambda^2\rho^2} + \delta \frac{\nu}{1 + \lambda^2\rho^2} = 0. \quad (6.3)$$

Thus $u_0(\rho)$ is a solution of the boundary value problem (6.2) and (6.3); it will depend on ω and δ .

In view of the linearity of the boundary value problem we may write

$$u_0 = b\omega u_1 + \delta u_2 \quad (6.4)$$

and split (6.2), (6.3) into the following two boundary value problems for u_1 and u_2 .

$$Lu_1 = \frac{(1 - \nu)\lambda\rho}{1 + \lambda^2\rho^2} - \frac{(1 + \nu)\lambda\rho}{(1 + \lambda^2\rho^2)^2}, \quad (6.5)$$

$$\rho = \pm 1: \quad \frac{du_1}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2} u_1 = \frac{-\nu\lambda}{1 + \lambda^2}, \quad (6.6)$$

$$Lu_2 = \frac{(1 + \nu)\lambda^2\rho}{(1 + \lambda^2\rho^2)^2}, \quad (6.7)$$

$$\rho = \pm 1: \quad \frac{du_2}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2} u_2 = \frac{-\nu}{1 + \lambda^2\rho^2}, \quad (6.8)$$

where L is the differential operator on the left side of (6.2).

The differential equation $Lu = 0$ is, in different notation, the equation derived by Sanders [5] for helicoidal shell problems in which the displacements are independent of θ . It is possible to reduce this equation to hypergeometric form in a number of different ways, but no use will be made of this possibility in what follows.

7. Influence coefficients. When the boundary value problems (6.5) to (6.8) have been solved, we shall have all quantities expressed in terms of known functions of ρ , and the constants ω and δ . Thus by (5.20) to (5.23) and (6.4),

$$(1 - \nu^2)n_r = b\omega \left[\frac{du_1}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2} u_1 + \frac{\nu\lambda\rho^2}{1 + \lambda^2\rho^2} \right] + \delta \left[\frac{du_2}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2} u_2 + \frac{\nu}{1 + \lambda^2\rho^2} \right], \quad (7.1)$$

$$(1 - \nu^2)n_\theta = b\omega \left[\nu \frac{du_1}{d\rho} + \frac{\lambda^2 \rho}{1 + \lambda^2 \rho^2} u_1 + \frac{\lambda \rho^2}{1 + \lambda^2 \rho^2} \right] + \delta \left[\nu \frac{du_2}{d\rho} + \frac{\lambda^2 \rho}{1 + \lambda^2 \rho^2} u_2 + \frac{1}{1 + \lambda^2 \rho^2} \right], \quad (7.2)$$

$$12(1 - \nu^2)m_{r\theta} = b\omega \left[\frac{-\nu\lambda}{1 + \lambda^2 \rho^2} \frac{du_1}{d\rho} - \frac{(3 - 2\nu)\lambda^3 \rho}{(1 + \lambda^2 \rho^2)^2} u_1 + \frac{(1 - \nu) - (2 - \nu)\lambda^2 \rho^2}{(1 + \lambda^2 \rho^2)^2} \right] + \delta \left[\frac{-\nu\lambda}{1 + \lambda^2 \rho^2} \frac{du_2}{d\rho} - \frac{(3 - 2\nu)\lambda^3 \rho}{(1 + \lambda^2 \rho^2)^2} u_2 + \frac{(1 - \nu)\lambda^3 \rho^2 - (2 - \nu)\lambda}{(1 + \lambda^2 \rho^2)^2} \right], \quad (7.3)$$

$$12(1 - \nu^2)m_{\theta r} = b\omega \left[\frac{-\lambda}{1 + \lambda^2 \rho^2} \frac{du_1}{d\rho} - \frac{(2 - \nu)\lambda^3 \rho}{(1 + \lambda^2 \rho^2)^2} u_1 + \frac{(1 - \nu) - \lambda^2 \rho^2}{(1 + \lambda^2 \rho^2)^2} \right] + \delta \left[\frac{-\lambda}{1 + \lambda^2 \rho^2} \frac{du_2}{d\rho} - \frac{(2 - \nu)\lambda^3 \rho}{(1 + \lambda^2 \rho^2)^2} u_2 + \frac{(1 - \nu)\lambda^3 \rho^2 - \lambda}{(1 + \lambda^2 \rho^2)^2} \right]. \quad (7.4)$$

When these relations are introduced into the force and torque conditions (5.25), (5.26) we have two relations between the force F and torque T on the one hand and the angle of twist ω and the axial extension δ on the other hand. These relations may be written in the form

$$F = (2Ehb\gamma_{F\delta})\delta + (\frac{2}{3}Ehb^2\lambda\gamma_{F\omega})\omega, \quad (7.5)$$

$$T = (\frac{2}{3}Ehb^3\lambda\gamma_{T\delta})\delta + (\frac{2}{3}Ehb^3\lambda^2\gamma_{T\omega} + \frac{2}{3}Gh^3b\gamma)\omega, \quad (7.6)$$

where the dimensionless influence coefficients γ are given by

$$\gamma_{F\delta} = \frac{1}{1 - \nu^2} \int_0^1 \left[\frac{\nu}{(1 + \lambda^2 \rho^2)^{1/2}} \frac{du_2}{d\rho} + \frac{\lambda^2 \rho u_2}{(1 + \lambda^2 \rho^2)^{3/2}} + \frac{1}{(1 + \lambda^2 \rho^2)^{3/2}} \right] d\rho, \quad (7.7)$$

$$\gamma_{F\omega} = \frac{3}{1 - \nu^2} \frac{1}{\lambda} \int_0^1 \left[\frac{\nu}{(1 + \lambda^2 \rho^2)^{1/2}} \frac{du_1}{d\rho} + \frac{\lambda^2 \rho}{(1 + \lambda^2 \rho^2)^{3/2}} u_1 + \frac{\lambda \rho^2}{(1 + \lambda^2 \rho^2)^{3/2}} \right] d\rho, \quad (7.8)$$

$$\gamma_{T\delta} = \frac{3}{1 - \nu^2} \int_0^1 \left[\frac{\nu \rho^2}{(1 + \lambda^2 \rho^2)^{1/2}} \frac{du_2}{d\rho} + \frac{\lambda^2 \rho^3}{(1 + \lambda^2 \rho^2)^{3/2}} u_2 + \frac{\rho^2}{(1 + \lambda^2 \rho^2)^{3/2}} \right] d\rho, \quad (7.9)$$

$$\gamma_{T\omega} = \frac{5}{1 - \nu^2} \frac{1}{\lambda} \int_0^1 \left[\frac{\nu \rho^2}{(1 + \lambda^2 \rho^2)^{1/2}} \frac{du_1}{d\rho} + \frac{\lambda^2 \rho^3}{(1 + \lambda^2 \rho^2)^{3/2}} u_1 + \frac{\lambda \rho^4}{(1 + \lambda^2 \rho^2)^{3/2}} \right] d\rho, \quad (7.10)$$

and finally

$$\gamma = \int_0^1 \frac{d\rho}{(1 + \lambda^2 \rho^2)^{7/2}} = \frac{1 + (4\lambda^2/3) + (8\lambda^4/15)}{(1 + \lambda^2)^{5/2}}. \quad (7.11)$$

In the integrals (7.7) to (7.11) use has been made of the fact that u_1 and u_2 are odd in ρ , so that the integrands are even and $\int_{-1}^1 = 2\int_0^1$.

It will be shown in Sec. 8 that when $\lambda = 0$ we have $\gamma_{F\delta} = \gamma_{F\omega} = \gamma_{T\delta} = \gamma_{T\omega} = \gamma = 1$, so that (7.5) and (7.6) become

$$F = 2Ehb, \quad T = \frac{2}{3}Gh^3b\omega,$$

corresponding to well-known results in the theory of extension and torsion of a flat plate by end loads.

A straightforward application of Green's formula, making use of the fact that u_1

and u_2 are solutions of (6.5) to (6.8), shows that $\gamma_{F\omega} = \gamma_{T\delta}$, as is in fact required by the reciprocal work theorem of elasticity.

In addition to the flexibility relations (7.5), (7.6), we have the associated inverse equations

$$\begin{aligned}\omega &= K_{\omega T}T + K_{\omega F}F \\ \delta &= K_{\delta T}T + K_{\delta F}F\end{aligned}\quad (7.12)$$

It is readily shown from (7.5) and (7.6) that

$$\left. \begin{aligned}K_{\omega T} &= \frac{3}{2Gh^3b[\gamma(\lambda) + (4Eb^2/15Gh^2)\lambda^2f(\lambda)]} \\ K_{\delta T} = K_{\omega F} &= \frac{-\lambda k(\lambda)}{2Gh^3[\gamma(\lambda) + (4Eb^2/15Gh^2)\lambda^2f(\lambda)]} \\ K_{\delta F} &= \frac{k_{\delta F}(\lambda) + (3Eb^2/5Gh^2)\lambda^2g(\lambda)}{2Ehb[\gamma(\lambda) + (4Eb^2/15Gh^2)\lambda^2f(\lambda)]}\end{aligned}\right\}, \quad (7.13)$$

where the abbreviations

$$\left. \begin{aligned}f(\lambda) &= \frac{9}{4}\gamma_{T\omega} - \frac{5}{4}\frac{\gamma_{F\omega}\gamma_{T\delta}}{\gamma_{F\delta}}, \\ k(\lambda) &= \frac{\gamma_{F\omega}}{\gamma_{F\delta}} = \frac{\gamma_{T\delta}}{\gamma_{F\delta}}, \\ k_{\delta F}(\lambda) &= \frac{\gamma}{\gamma_{F\delta}}, \quad g(\lambda) = \frac{\gamma_{T\omega}}{\gamma_{F\delta}}\end{aligned}\right\} \quad (7.14)$$

have been used.

In the absence of axial forces, the torque T and twist ω are related by

$$T = I\omega,$$

where the torsional rigidity I is derived from (7.12) and (7.13) as

$$I = \frac{1}{K_{\omega T}} = \frac{2}{3}Gh^3b\left[\gamma(\lambda) + \frac{4}{15}\frac{E}{G}\frac{b^2}{h^2}\lambda^2f(\lambda)\right]. \quad (7.15)$$

In the absence of torque the axial force F and the relative extension δ are related by

$$F = K\delta,$$

where the axial stiffness K is given by

$$K = \frac{1}{K_{\delta F}} = 2Ehb \frac{\gamma(\lambda) + (4Eb^2/15Gh^2)\lambda^2f(\lambda)}{k_{\delta F}(\lambda) + (3Eb^2/5Gh^2)\lambda^2g(\lambda)}. \quad (7.16)$$

In the following section the first three terms in the power series expansions for $\gamma_{F\delta}(\lambda)$, $\gamma_{F\omega}(\lambda)$, $\gamma_{T\omega}(\lambda)$, $\gamma(\lambda)$, $f(\lambda)$, $k(\lambda)$, $k_{\delta F}(\lambda)$, and $g(\lambda)$ are obtained by perturbation methods.

8. Perturbation solutions. All quantities of physical interest have been expressed in terms of the solutions of the two boundary value problems (6.5) to (6.8). Inspection of these problems indicates that $u_1(\rho, \lambda)$ is odd in λ and $u_2(\rho, \lambda)$ is even in λ , while both

are odd functions of ρ . For sufficiently small λ , we obtain solutions in the form

$$\begin{aligned} u_1(\rho, \lambda) &= \lambda u_1^{(1)}(\rho) + \lambda^3 u_1^{(3)}(\rho) + \cdots, \\ u_2(\rho, \lambda) &= u_2^{(0)}(\rho) + \lambda^2 u_2^{(2)}(\rho) + \cdots. \end{aligned} \quad (8.1)$$

Introduction of these assumptions into the boundary value problems (6.5) to (6.8) leads to a sequence of boundary value problems for the $u_i^{(j)}(\rho)$. These may be solved successively by repeated integration. The results of such calculations are

$$\begin{aligned} u_1 &= -\frac{\nu}{3} \rho^3 \lambda + \left[-\frac{1-\nu^2}{4} \rho + \left(\frac{1}{20} + \frac{\nu}{5} + \frac{\nu^2}{60} \right) \rho^5 \right] \lambda^3 \\ &+ \left[(1-\nu^2) \frac{9+5\nu}{36} \rho + \frac{(1-\nu^2)(1+\nu)}{24} \rho^3 - \frac{(135+383\nu+57\nu^2+\nu^3)}{2520} \rho^7 \right] \lambda^5 \\ &+ O(\lambda^7), \end{aligned} \quad (8.2)$$

$$\begin{aligned} u_2 &= -\nu \rho + \left[-\frac{1-\nu^2}{2} \rho + \frac{(1+\nu)^2}{6} \rho^3 \right] \lambda^2 + \left[\frac{3}{8} (1+\nu)(1-\nu^2) \rho \right. \\ &+ \frac{1}{12} (1-\nu^2)(1+\nu) \rho^3 - \frac{(1+\nu)^2(15+\nu)}{120} \rho^5 \left. \right] \lambda^4 \\ &+ \left[\frac{(1+\nu)^2(1-\nu)(31-37\nu)}{144} \rho - \frac{(1-\nu^2)(1+\nu)^2}{16} \rho^3 \right. \\ &- \frac{(1-\nu^2)(1+\nu)(15+\nu)}{240} \rho^5 + \frac{(1-\nu^2)(515+60\nu+\nu^2)}{5040} \rho^7 \left. \right] \lambda^6 \\ &+ O(\lambda^8). \end{aligned} \quad (8.3)$$

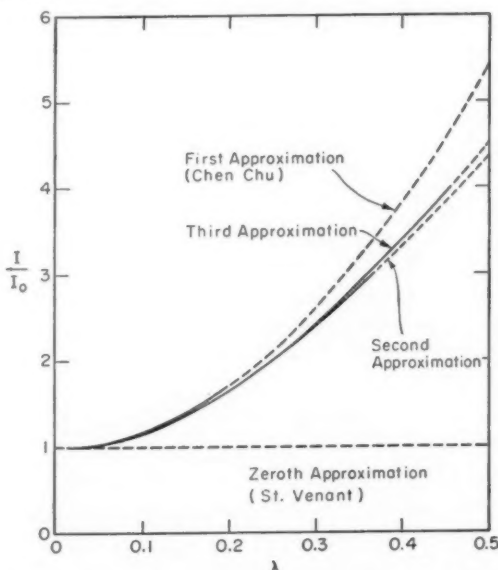


FIG. 3 Torsional rigidity ratio vs. pretwist parameter for $\nu = 1/3$ and $2b/h = 10$

Introduction of the perturbation solutions (8.2) and (8.3) into the integrals (7.7) to (7.10) gives the power series expansions in λ of $\gamma_{F\delta}$, $\gamma_{F\omega} = \gamma_{T\delta}$, and $\gamma_{T\omega}$; $\gamma(\lambda)$ may be directly expanded from (7.11). These expansions are

$$\left. \begin{aligned} \gamma_{F\delta} &= 1 - \frac{3+4\nu}{6}\lambda^2 + \frac{29+88\nu+56\nu^2}{120}\lambda^4 + O(\lambda^6), \\ \gamma_{F\omega} = \gamma_{T\delta} &= 1 - \frac{9+8\nu}{10}\lambda^2 + \frac{161+304\nu+152\nu^2}{280}\lambda^4 + O(\lambda^6), \\ \gamma_{T\omega} &= 1 - \frac{45+20\nu}{42}\lambda^2 + \frac{531+504\nu+232\nu^2}{648}\lambda^4 + O(\lambda^6), \\ \gamma &= 1 - \frac{7}{6}\lambda^2 + \frac{63}{40}\lambda^4 + O(\lambda^6). \end{aligned} \right\} \quad (8.4)$$

These in turn provide the expansions of $f(\lambda)$, $k(\lambda)$, $k_{\delta F}(\lambda)$ and $g(\lambda)$ defined in (7.14).

$$\left. \begin{aligned} f &= 1 - \frac{33-4\nu}{42}\lambda^2 + \frac{427-152\nu+8\nu^2}{840}\lambda^4 + O(\lambda^6), \\ k &= 1 - \frac{6+2\nu}{15}\lambda^2 + \frac{42+6\nu-4\nu^2}{315}\lambda^4 + O(\lambda^6), \\ k_{\delta F} &= 1 - \frac{2}{3}(1-\nu)\lambda^2 + \frac{45-38\nu-\nu^2}{45}\lambda^4 + O(\lambda^6), \\ g &= 1 - \frac{12-4\nu}{21}\lambda^2 + \frac{828-486\nu+52\nu^2}{2835}\lambda^4 + O(\lambda^6). \end{aligned} \right\} \quad (8.5)$$

In particular the torsional rigidity I of formula (7.15) takes the form

$$\begin{aligned} I &= I_0 \left[\left(1 - \frac{7}{6}\lambda^2 + \frac{63}{40}\lambda^4 + \dots \right) \right. \\ &\quad \left. + \frac{4}{15} \frac{E}{G} \frac{b^2}{h^2} \lambda^2 \left(1 - \frac{33-4\nu}{42}\lambda^2 + \frac{427-152\nu+8\nu^2}{840}\lambda^4 + \dots \right) \right], \end{aligned} \quad (8.6)$$

where

$$I_0 = I_{\lambda=0} = \frac{2}{3} G h^3 b \quad (8.7)$$

is the St. Venant torsional rigidity of a flat plate of thickness h and width $2b$. If only the first term in each power series in parenthesis in (8.6) is retained, we obtain the Chen Chu approximation [2]

$$I \cong I_0 \left(1 + \frac{4}{15} \frac{E}{G} \frac{b^2}{h^2} \lambda^2 \right) \quad (8.8)$$

indicating the increase in torsional stiffness for small pretwist. Figure 3 compares the Chu approximation (8.8) with the second approximation (retaining λ^2 terms in the two power series) and the third approximation (retaining λ^4 terms).

Corresponding results for the axial stiffness K of (7.16) are obtained by inserting the power series for γ , f , $k_{\delta F}$ and g into (7.16). There follows

$$\frac{K}{K_0} = \left[\left(1 - \frac{7}{6} \lambda^2 + \frac{63}{40} \lambda^4 + \dots \right) + \frac{4}{15} \frac{E}{G} \frac{b^2}{h^2} \lambda^2 \left(1 - \frac{33 - 4\nu}{42} \lambda^2 + \frac{427 - 152\nu + 8\nu^2}{840} \lambda^4 + \dots \right) \right] \left[\left(1 - \frac{2 - 2\nu}{3} \lambda^2 + \frac{45 - 38\nu - \nu^2}{45} \lambda^4 + \dots \right) + \frac{3}{5} \frac{E}{G} \frac{b^2}{h^2} \lambda^2 \left(1 - \frac{12 - 4\nu}{21} \lambda^2 + \frac{828 - 486\nu + 52\nu^2}{2835} \lambda^4 + \dots \right) \right]^{-1}, \quad (8.9)$$

where

$$K_0 = K_{\lambda=0} = 2Ehb \quad (8.10)$$

is the axial stiffness of a flat plate according to plane stress. If only the first term is retained in each of the four power series in parentheses in (8.9) there follows what may be considered as an analogue of the Chu approximation;

$$K \cong K_0 \frac{1 + (4Eb^2/15Gh^2)\lambda^2}{1 + (3Eb^2/5Gh^2)\lambda^2}. \quad (8.11)$$

Figure 4 compares the first approximation (8.11) with the second and third approximations obtained by retaining the λ^2 and λ^4 terms, respectively, in the four power series in parentheses in (8.9).

In addition to the influence coefficients, the stress resultants and couples themselves may be calculated by introducing the expansions (8.2) and (8.3) into the expression

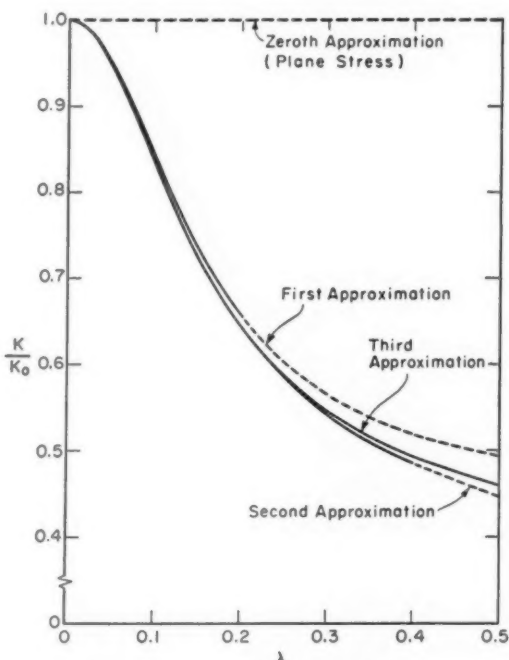


FIG. 4 Extensional stiffness ratio vs. pretwist parameter for $\nu = 1/3$ and $2b/h = 10$

(7.1) to (7.4) for n_r , n_θ , $m_{r\theta}$, and $m_{\theta r}$. We obtain

$$n_r = b\omega \left[-\frac{1}{4}(1 - \rho^4)\lambda^3 + \left(\frac{9+5\nu}{36} + \frac{1-\nu}{8}\rho^2 - \frac{27+\nu}{72}\rho^6 \right) \lambda^5 + \dots \right] \\ + \delta \left[-\frac{1}{2}(1 - \rho^2)\lambda^2 + \left(\frac{3+3\nu}{8} + \frac{1-\nu}{4}\rho^2 - \frac{5+\nu}{8}\rho^4 \right) \lambda^4 + \dots \right], \quad (8.12)$$

$$n_\theta = b\omega \left[\rho^2\lambda - \left(\frac{\nu}{4} + \frac{12+\nu}{12}\rho^4 \right) \lambda^3 + \left(\frac{9\nu+5\nu^2}{36} - \frac{2-\nu-\nu^2}{8}\rho^2 \right. \right. \\ \left. \left. + \frac{378+57\nu+\nu^2}{360}\rho^6 \right) \lambda^5 + \dots \right] + \delta \left[1 - \left(\frac{\nu}{2} + \frac{2+\nu}{2}\rho^2 \right) \lambda^2 \right. \\ \left. + \left(\frac{3\nu+3\nu^2}{8} - \frac{2-\nu-\nu^2}{4}\rho^2 + \frac{28+17\nu+\nu^2}{24}\rho^4 \right) \lambda^4 + \dots \right], \quad (8.13)$$

$$12(1+\nu)m_{r\theta} = b\omega \left[1 - (4+\nu)\rho^2\lambda^2 + \left(\frac{\nu+\nu^2}{4} + \frac{84+33\nu+\nu^2}{12}\rho^4 \right) \lambda^4 + \dots \right] \\ + \delta \left[-(2+\nu)\lambda + \left(\frac{\nu+\nu^2}{2} + \frac{10+9\nu+\nu^2}{2}\rho^2 \right) \lambda^3 + \left(-\frac{3\nu+6\nu^2+3\nu^3}{8} \right. \right. \\ \left. \left. + \frac{6-\nu-8\nu^2-\nu^3}{4}\rho^2 - \frac{204+217\nu+38\nu^2+\nu^3}{24}\rho^4 \right) \lambda^5 + \dots \right], \quad (8.14)$$

$$12(1+\nu)m_{\theta r} = b\omega \left[1 - 3\rho^2\lambda^2 + \left(\frac{1+\nu}{4} + \frac{57+5\nu}{12}\rho^4 \right) \lambda^4 + \dots \right] \\ + \delta \left[-\lambda + \left(\frac{1+\nu}{2} + \frac{5+3\nu}{2}\rho^2 \right) \lambda^3 + \left(-\frac{3+6\nu+3\nu^2}{8} \right. \right. \\ \left. \left. + \frac{1-2\nu-3\nu^2}{4}\rho^2 - \frac{101+82\nu+5\nu^2}{24}\rho^4 \right) \lambda^5 + \dots \right]. \quad (8.15)$$

If (7.12), (7.13) and (8.5) are used to express ω and δ in terms of F and T in (8.12) to (8.15), the expression for stress resultants and couples are obtained in terms of F and T . The corresponding expressions are lengthy and will not be given here.

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WAVE PROPAGATION IN A COAXIAL SYSTEM*

BY

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Abstract. A solution is obtained for the problem of the propagation of electromagnetic waves in a semi-infinite flanged coaxial line with an infinite center conductor, in terms of an infinite set of coefficients which are determined by an infinite set of linear equations. The solution is discussed, in detail, in limiting cases which illustrate properties both of a thin vertical antenna on a plane perfectly conducting earth, and of a thick antenna fed by a low impedance line. Numerical results are given in these cases. The possibility of a solution for any excitation frequency is also discussed.

Introduction. The use of an antenna for the purpose of radiating electromagnetic energy must involve the generation of this energy and its passage to the antenna from the generator along a transmission line. There is, however, very little theoretical work published in which an antenna is considered with its means of excitation. It might appear to be an advantage to examine the radiating system in isolation, and then to regard it as an impedance lumped at the end of a transmission line: the magnitude of such an impedance depends, however, on the transmission line parameters. It is therefore of interest to find whether there is any range of parameters for which the line and the antenna are substantially independent. The work done on this subject by King and others [3] implies that as we reduce the spacing of the transmission line to zero we may extrapolate from a series of experimental measurements of physical quantities to the limiting case of zero spacing; this value may be identified with theoretical results obtained by assuming, in the isolated radiating system, a delta-function excitation (sometimes called a "slice" generator) at the driving point of the antenna. This statement needs qualification, since the question of the existence of the physical quantity in the limit of zero spacing was not considered by King. We find that this limit does not indeed exist. Accordingly, in this paper we are concerned with an idealization of the cylindrical antenna problem in which the complete transmission circuit may be examined mathematically. As an approach to the problem of the antenna of finite length, we shall consider a semi-infinite flanged coaxial line with its center conductor extended to an infinite length. A signal set up at one end of the line is partly reflected at the open end and partly radiated into the half-space outside the line. Since we are interested in antennae, we may describe the radiating system as an infinite vertical antenna of circular cross-section, standing on a horizontal, perfectly conducting ground of infinite extent, with the exciting signal fed in at the bottom.

We shall further simplify the system by considering the case in which the field is radially symmetric about the axis of the cylinder, with no magnetic field component parallel to this axis. The field components are related by Maxwell's equations; in this

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problem all the non-zero field components may be written down in terms of the transverse component of the magnetic field tangential to the surface of the cylinder, and this component satisfies the scalar wave equation. The normal derivatives of the magnetic field vanish on all the perfectly conducting surfaces. The field components in the half-space are related to the normal derivative of the magnetic field in the terminal plane of the coaxial line: this relationship may be found by a method in which cosine transforms are used. Since the field within the line may be expressed in terms of an infinite set of discrete modes (Marcuvitz [4]), it is possible by matching orthogonal components across the gap at the open end of the line to set up an infinite set of linear equations involving an infinite set of unknown coefficients. Each coefficient is related simply to the amplitude of the corresponding mode in the line; in turn, the set of coefficients determines all the field components completely and uniquely. To simplify the analysis, the free-space propagation constant is taken to have a small negative imaginary part which is later taken to be zero. The resulting solution is shown to satisfy the conditions of the problem.

The solution of the infinite set of equations is simplified when the spacing of the coaxial line is small compared to the wavelength of the exciting signal. This simplifying condition is valid either when the outer diameter of the line is a small fraction of the wavelength or when the spacing between the conductors in the line is a small fraction of the line diameter. With this simplification, we find that the field within the line is very nearly that in an open-circuited line, and this approximate field leads us to a good value both for the admittance of the radiating system and for the energy density at large distances from the antenna. The effect of the simplification is to give us results not only for the thick cylindrical antenna with a small line spacing but also for the thin antenna with a physically plausible method of excitation.

It should be emphasized that the results are appropriate to an idealized antenna of infinite length. The energy distribution may not be related to that of a finite antenna. On the other hand, the admittance of the infinite antenna is the limiting value of the admittance of long antennae.

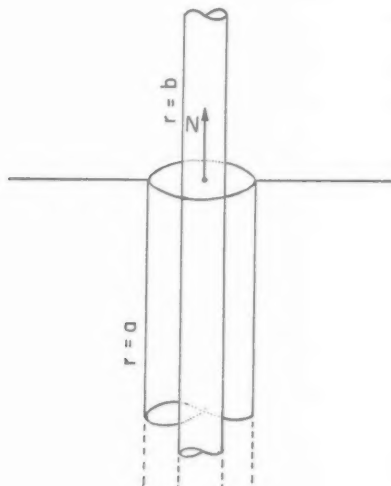


FIG. 1

Results associated with a similar geometry without a flange have been given by Marcuvitz [4]. It is implied that these results were obtained by a method involving the use of the Wiener-Hopf technique. In the present problem a similar technique may well be practical, but the author has not yet made a comparison.

I. The geometry of the system to be considered is shown in Fig. 1. Cylindrical co-ordinates (r, θ, z) are used. The axis of an infinite, circular, conducting cylinder of radius $b > 0$ is taken to be the z -axis. In the region $z < 0$ this cylinder is surrounded by a semi-infinite coaxial conducting surface of radius a ($a > b$), and this surface, $r = a$, $z < 0$, is terminated in the plane $z = 0$ by a perfectly conducting plane in the region $r > a$, $z = 0$. A time dependence $\exp(i\omega t)$ is assumed throughout, and rationalized m.k.s. units are used.

We are to consider the electromagnetic field in the region bounded by the conducting surfaces corresponding to a combination of axially symmetric transverse magnetic modes in the line region $a > r > b$, $z < 0$. This type of field is independent of the co-ordinate θ , and it has the magnetic field components B_r and B_z both zero. From Maxwell's equations, it follows that the electric field components are related to the magnetic field component B_θ by the equations,

$$\begin{aligned}\frac{ik^2}{\omega} E_z &= \frac{1}{r} \frac{\partial}{\partial r} (rB_\theta), \\ \frac{ik^2}{\omega} E_r &= -\frac{\partial B_\theta}{\partial z}, \\ E_\theta &= 0,\end{aligned}\tag{1}$$

where k is taken with a small negative imaginary part ($k = k_r - ik_i$, $k_i, k_r > 0$), and B_θ must satisfy the equation,

$$\left(\frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + k^2 - \frac{1}{r^2} \right) B_\theta = 0.\tag{2}$$

Since the tangential component of the electric field vanishes at a perfectly conducting surface, B_θ must satisfy the boundary conditions

$$\frac{\partial}{\partial r} (rB_\theta) = 0 \quad \text{on } r = b,\tag{3}$$

$$\frac{\partial}{\partial r} (rB_\theta) = 0 \quad \text{on } r = a, \quad z < 0,\tag{4}$$

$$\frac{\partial}{\partial z} (B_\theta) = 0 \quad \text{on } z = 0, \quad r > a.\tag{5}$$

We shall assume the total field to be made up of an incident wave in the dominant mode within the coaxial line, with magnetic component $B_{\theta_+}(r, z)$ and a scattered field with magnetic components $B_{\theta_+}(r, z)$ for $z > 0$, and $B_{\theta_-}(r, z)$ for $z < 0$. B_{θ_+} is defined by the equation,

$$B_{\theta_+} = \exp(-ikz)/r \quad \text{for } a > r > b, \quad z < 0.\tag{6}$$

With $k_i > 0$, the radiation condition at infinity is satisfied if we take B_{θ_-} to be of backward-travelling wave form within the line, and B_{θ_+} to have a wave form travelling away

from the open end of the line. Thus, in the half-space $z > 0$ the behavior of $B_{\theta+}$ is given by the relation

$$RB_{\theta+} \sim f(\varphi) \exp(-ikR) \quad \text{as } R \rightarrow \infty, \quad 0 < \varphi < \pi/2, \quad (7)$$

where $R = (r^2 + z^2)^{1/2}$, $\varphi = \tan^{-1}(r/z)$, and $f(\varphi)$ is a bounded function independent of R . At the edge $r = a$, $z = 0$, the condition on the total field which ensures the integrability of the edge current and hence the uniqueness of the solution [2, 5] is that $B_{\theta} = \text{constant} - O(\sigma^{2/3})$, and $\partial B_{\theta}/\partial z = O(\sigma^{-1/2})$ as the distance σ of the point of observation to the edge tends to zero. The problem is now completely and uniquely specified.

II. To find the field in the half-space $z > 0$, we define a Fourier cosine transform for all $r > b$ by the equation,

$$F(p, r) = \int_0^{\infty} B_{\theta+}(r, z) \cos pz \, dz. \quad (8)$$

The absolute value of this integral for large values of z is found from Eq. 7 to contain the factor $\exp -k_i (r^2 + z^2)^{1/2}$, and this factor ensures the absolute convergence of the integral. The inverse of Eq. 8 is given by

$$B_{\theta+}(r, z) = 2 \int_0^{\infty} F(p, r) \cos pz \, dp/\pi, \quad (9)$$

for all $r > b$.

If we multiply Eq. 2 by $\cos pz$ and integrate with respect to z in the range $z > 0$, we find that

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + K^2 - \frac{1}{r^2} \right) F(p, r) = \left(\frac{\partial B}{\partial z} \right)_{z=0}$$

so that for $r > a$, from Eq. 5, $F(p, r)$ satisfies the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + K^2 - \frac{1}{r^2} \right) F(p, r) = 0, \quad (10)$$

and for $a > r > b$, writing $L(r)$ for $(\partial B_{\theta}/\partial z)_{z=0}$, $F(p, r)$ satisfies the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + K^2 - \frac{1}{r^2} \right) F(p, r) = L(r). \quad (11)$$

Here $K = (k^2 - p^2)^{1/2}$; that branch of the square root is taken for which when $k_i = 0$, K is real and positive when p is real and $|p| < k$, and for which iK is real and positive when p is real and $|p| > k$.

From Eq. 3, $\partial[rF(p, r)]/\partial r$ must vanish at $r = b$. On the surface $r = a$, B_{θ} and $\partial B_{\theta}/\partial z$ are functions of integrable square for all $z > 0$: the continuity of the field components B_{θ} and E_z across this surface therefore ensures the continuity of the functions $F(p, r)$ and $\partial[rF(p, r)]/\partial r$ at $r = a$. From Eqs. 7 and 8 it follows that as $r \rightarrow \infty$

$$F(p, r) \sim \int_0^{\infty} f(\varphi) \exp(-ikR) \cos pz \, dz/R.$$

For large values of r this integral may be evaluated by the method of steepest descents: the calculation shows that

$$F(p, r) \sim \psi(p) \exp(-ikr)/r^{1/2},$$

where $\psi(p)$ is a function of p only.

The solution of Eq. 10 which shows this behavior for large values of r is a multiple of $H_1(Kr)$; $H_1(Kr)$ is that solution of Bessel's equation for which as $r \rightarrow \infty$ for real or imaginary values of K

$$(Kr)^{1/2}H_1(Kr) \sim \exp[-i(Kr - 3\pi/4)][2/\pi]^{1/2}.$$

When K is real this is the Hankel function of $H_1^{(2)}(Kr)$. Similarly, $H_0(Kr)$ is a zero order Bessel function which for real values of K is to be identified with the Hankel function $H_0^{(2)}(Kr)$.

The solutions of Eqs. 10 and 11 which satisfy the required boundary and continuity conditions for $F(p, r)$ and $\partial[rF(p, r)]/\partial r$ are, for $r > a$.

$$F(p, r) = \pi H_1(Kr) \int_b^a \rho L(\rho) [J_1(K\rho) Y_0(Kb) - Y_1(K\rho) J_0(Kb)] d\rho / 2H_0(Kb), \quad (12)$$

and for $b < r < a$,

$$\begin{aligned} 2F(p, r) [J_0(Ka) Y_0(Kb) - Y_0(Ka) J_0(Kb)] / \pi = & [J_1(Kr) Y_0(Kb) - Y_0(Ka) J_0(Kb)] \\ & \cdot \int_b^a \rho L(\rho) H_0(Ka) [J_1(K\rho) Y_0(Kb) - Y_1(K\rho) J_0(Kb)] d\rho / H_0(Kb) \\ & - [J_1(Kr) Y_0(Kb) - Y_1(Kr) J_0(Kb)] \\ & \cdot \int_r^a \rho L(\rho) [J_1(K\rho) Y_0(Ka) - Y_1(K\rho) J_0(Ka)] d\rho \\ & - [J_1(Kr) Y_0(Ka) - Y_1(Kr) J_0(Ka)] \\ & \cdot \int_b^r \rho L(\rho) [J_1(K\rho) Y_0(Kb) - Y_1(K\rho) J_0(Kb)] d\rho. \end{aligned} \quad (13)$$

These equations determine the cosine transforms of the magnetic field in the half-space in terms of the function $L(r)$ which is proportional to the radial electric field in the gap $z = 0$, $a > r > b$.

From Eqs. 9 and 13 we may write down the total magnetic field in the gap. This is

$$\begin{aligned} B_{\theta+}(r, \theta) = \int_0^\infty d\rho \left\{ \frac{J_1(Kr) Y_0(Kb) - Y_1(Kr) J_0(Kb)}{J_0(Ka) Y_0(Kb) - Y_0(Ka) J_0(Kb)} \right\} \frac{H_0(Ka)}{H_0(Kb)} \\ \cdot \int_b^a \rho L(\rho) [J_1(K\rho) Y_0(Kb) - Y_1(K\rho) J_0(Kb)] d\rho - \frac{J_1(Kr) Y_0(Ka) - Y_1(Kr) J_0(Ka)}{J_0(Ka) Y_0(Kb) - Y_0(Ka) J_0(Kb)} \\ \cdot \int_b^r \rho L(\rho) [J_1(K\rho) Y_0(Kb) - Y_1(K\rho) J_0(Kb)] d\rho - \frac{J_1(Kr) Y_0(Kb) - Y_1(Kr) J_0(Kb)}{J_0(Ka) Y_0(Kb) - Y_0(Ka) J_0(Kb)} \\ \cdot \int_r^a \rho L(\rho) [J_1(K\rho) Y_0(Ka) - Y_1(K\rho) J_0(Ka)] d\rho. \end{aligned} \quad (14)$$

III. In the region $z < 0$, $a > r > b$, the field can be described in terms of a set of axially symmetric transverse magnetic modes. These modes are associated with the set of functions $r^{1/2} \varphi_n(r)$, orthogonal in $a > r > b$, where

$$\varphi_0(r) = 1/r,$$

and

$$\varphi_n(r) = J_1(K_n r) Y_0(K_n a) - Y_1(K_n r) J_0(K_n a),$$

for integer values of $n \geq 1$. When $n \geq 1$, $K = K_n$ is the n th positive zero in order of magnitude of the function $J_0(Ka)Y_0(Kb) - Y_0(Ka)J_0(Kb)$, and $K_0 = 0$. For all $n > 1$, $(a - b)K_n \sim n\pi$ [1]. If we put p_n to be that value of p which corresponds to the value $K = K_n$, then the magnetic field component of any axially symmetric TM field within the line is a linear combination of modes of the form $\varphi_n(r) \exp(\pm ip_n z)$.

We have assumed that an incident dominant mode $B_{\theta+}$ of unit amplitude travels towards the open end of the line. If R_n is the reflexion coefficient associated with the n th mode reflected at the plane $z = 0$, for an incident dominant mode of unit amplitude, then we may write the reflected field in the form

$$B_{\theta-} = \sum_{n=0}^{\infty} R_n \varphi_n(r) \exp(ip_n z), \quad (15)$$

so that

$$\delta B_{\theta-} / \delta z = \sum_{n=0}^{\infty} ip_n R_n \varphi_n(r) \exp(ip_n z). \quad (16)$$

By matching both the total magnetic field and its normal derivative across the open end of the line we find that

$$B_{\theta+}(r, 0) = \sum_{n=0}^{\infty} (R_n + \delta_{0n}) \varphi_n(r), \quad (17)$$

and

$$L(r) = \sum_{n=0}^{\infty} ip_n (R_n - \delta_{0n}) \varphi_n(r), \quad (18)$$

where $\delta_{mn} = 0$ for $m \neq n$, and $\delta_{mn} = 1$ for all integer values of m . Both the functions $r^{1/2}L(r)$ and $r^{1/2}B_{\theta+}(r, 0)$ are functions of integrable square in $a > r > b$, so that they may be expanded in orthogonal series of the functions $\varphi_n(r)$.

Thus, if in $a > r > b$

$$L(r) = \sum_{n=0}^{\infty} a_n \varphi_n(r),$$

then

$$a_n N_n = \int_b^a r L(r) \varphi_n(r) dr, \quad (19)$$

where

$$N_n = \int_b^a r [\varphi_n(r)]^2 dr.$$

$$\begin{cases} N_0 = \ln(a/b), \\ N_n = 2[1 - [J_0(K_n a)/J_0(K_n b)]^2] / (\pi K_n^2) \end{cases}.$$

We can see therefore from Eqs. 18 and 19 that

$$a_n = ip_n (R_n - \delta_{0n}), \quad (20)$$

and from Eq. 17, that

$$\int_b^a r B_{\theta+}(r, 0) \varphi_n(r) dr = N_n (R_n + \delta_{0n}). \quad (21)$$

Since we are able to write $B_{\theta+}(r, 0)$ in terms of the coefficients a_n , Eq. 21 may be manipulated to represent an infinite set of equations linear in the a_n whose solution will determine the field throughout the system.

Thus, from Eqs. 14 and 19, assuming that the required changes in the orders of integration and summation are permissible, it follows that

$$k \int_b^a r B_{\theta+}(r, 0) \varphi_m(r) dr = \sum_{n=0}^{\infty} C_{mn} a_n, \quad (22)$$

where

$$C_{mn} = k \int_0^{\infty} dp \left\{ K^2 a^2 \frac{H_0(Ka) \varphi_m(a) \varphi_n(a)}{H_0(Kb) (K^2 - K_n^2) (K^2 - K_m^2)} \right. \\ \left. \cdot [J_0(Ka) Y_0(Kb) - Y_0(Ka) J_0(Kb)] + \frac{2N_n \delta_{nm}}{\pi K_{\Delta}^2 - \pi K_n^2} \right\}. \quad (23)$$

It follows from Eqs. 21 and 22 that the infinite set of equations relating the coefficients a_n is

$$2 \delta_{0m} N_m + a_n N_m / (ip_m) = \sum_{n=0}^{\infty} C_{mn} a_n / k, \quad (24)$$

for all integer $n \geq 0$.

IV. In the complex p -plane the integrand defined in Eq. 23 has for singularities only branch points at $p = \pm k_0$. It is clear that the path of integration for the integral C_{mn} must be deformed into an equivalent contour: this is necessary in order to avoid passing through the branch point $p = \pm k$ in the limiting case when $k_i = 0$. Thus, in the neighborhood of this branch point we can deform the path of integration into a semi-circular arc of small radius δ in the upper half p -plane. To examine the contribution to the integral C_{mn} from this arc we put $p = k + \delta \exp i\varphi$, $0 \leq \varphi \leq \pi$, where we can now take k_i to be zero. Since as $\delta \rightarrow 0$ the corresponding form for K is

$$K = (2k \delta)^{1/2} \exp i(\theta - \pi)/2 [1 + 0(\delta)],$$

we can show that

$$\pi [J_0(Ka) Y_0(Kb) - Y_0(Ka) J_0(Kb)] / 2 = \ln b/a + 0[\delta(a - b)];$$

it follows that for $m = n = 0$, when $N_0 = \ln a/b$, the absolute value of the integrand for small values of δ is $O(1)$, and the contribution to the integral C_{00} is $O(\delta)$. For m or n not zero the contribution may be shown to be even smaller in magnitude. We may therefore write C_{mn} in the form of a line integral, this being the limiting case for $\delta = 0$. The question of the convergence of this integral is considered in the Appendix. It is shown there that C_{mn} is uniformly convergent with respect to the parameters a and b in the range $a > b > 0$. Since we have defined the branch of k so that for p real, $0 < p < k$, k is real, and for $p > k$, $k = -i\mu$ where μ is real, it follows that

$$\frac{C_{mn}}{k} = \int_0^k dp \left\{ K^2 a^2 \frac{H_0(Ka) \varphi_m(a) \varphi_n(a)}{H_0(Kb) [K^2 - K_m^2] (K^2 - K_n^2)} \right. \\ \left. \cdot [J_0(Ka) Y_0(Kb) - Y_0(Ka) J_0(Kb)] + \frac{2}{\pi} \frac{N_n \delta_{nm}}{K^2 - K_m^2} \right\} \\ + \frac{2}{\pi} \int_0^{\infty} d\mu \left\{ \mu^2 a^2 \frac{K_0(\mu a)}{K_0(\mu b)} \frac{\varphi_m(a) \varphi_n(a)}{(\mu^2 + K_m^2)(\mu^2 + K_n^2)} \right. \\ \left. \cdot [I_0(\mu a) K_0(\mu b) - I_0(\mu b) K_0(\mu a)] - \frac{N_n \varphi_{mn}}{K_m^2 + \mu^2} \right\}, \quad (25)$$

where $K = (k^2 - p^2)^{1/2}$, $\mu = (p^2 - k^2)^{1/2}$, and $K_0(z)$, $I_0(z)$ are modified Bessel functions [6].

V. The first quantity of physical interest to be considered is the energy density at large distances from the antenna. From Eqs. 9 and 12 it may be seen that for $r > a$,

$$B_{\theta+}(r, z) = \int_0^\infty \cos pz H_1(kr) \int_b^a L(\rho) [J_1(Kr) Y_0(Kb) - Y_1(Kr) J_0(Kb)] d\rho d\rho / H_0(Kb),$$

so that as $r \rightarrow \infty$, since the integrand is an even function of p , we have that

$$B_{\theta+}(r, z) \sim \text{const.} \int_{-\infty}^\infty \frac{dp \exp(ipz - iKr)}{(Kr)^{1/2} H_0(Kb)} \cdot \int_b^a \rho L(\rho) [J_1(Kr) Y_0(Kb) - Y_1(Kr) J_0(Kb)] d\rho. \quad (26)$$

This integral may be evaluated by the method of steepest descents. By putting $R = (r^2 + z^2)^{1/2}$, $\varphi = \tan^{-1} r/z$ with $0 < \varphi < \pi/2$, we find that the saddle point is at $p = -k \cos \varphi$, corresponding to the value $k = k \sin \varphi$. The path of integration is deformed, without changing the value of the integral, into the steepest descent path, which at the saddle-point makes an angle of $\pi/4$ with the line joining the branch points. The value of $B_{\theta+}$ for large values of R can now be given. This is

$$B_{\theta+} \sim \text{const.} \frac{\exp(-ikR)}{kRH_0^{(2)}(kb \sin \varphi)} \int_b^a \rho L(\rho) [J_1(k\rho \sin \varphi) Y_0(kb \sin \varphi) - Y_1(k\rho \sin \varphi) J_0(kb \sin \varphi)] d\rho, \quad (27)$$

and from Eq. 19 after performing the integration, we find that

$$B_{\theta+} \sim \frac{\text{const.} \sin \varphi \exp(-ikR)}{kRH_0^{(2)}(kb \sin \varphi)} [J_0(ka \sin \varphi) Y_0(kb \sin \varphi) - Y_0(kb \sin \varphi) J_0(ka \sin \varphi)] \sum_0^\infty \frac{a_n a_{\varphi_n}(a)}{k^2 \sin^2 \varphi - K_n^2}. \quad (28)$$

The energy density at large distances from the open end of the line can now be found from the complex Poynting vector, and it follows that the power flow P in the radial direction at large distances is given by

$$P = \text{const.} \frac{\sin^2 \varphi \{J_0(Ka \sin \varphi) Y_0(Kb \sin \varphi) - Y_0(Kb \sin \varphi) J_0(Ka \sin \varphi)\}^2}{R^2 \{[J_0(Kb \sin \varphi)]^2 + [Y_0(Kb \sin \varphi)]^2\}} \cdot \left\{ \sum_0^\infty \frac{a_n a_{\varphi_n}(a)}{k^2 \sin^2 \varphi - K_n^2} \right\}^2. \quad (29)$$

The second quantity which is of interest to us is the admittance of the antenna, regarded as a termination of the transmission line. If we go far enough along the line for higher order modes to be practically damped and then carry out standing wave measurements, we may determine the dominant mode reflexion coefficient R_0 . The terminal admittance Y_1 associated with this coefficient R_0 is given by the equation

$$Y_1 = Y_0(R_0 + 1)(R_0 - 1)^{-1}, \quad (30)$$

where the characteristic admittance of the line $Y_0 = 2\pi\omega\epsilon_0/k \ln(a/b)$. From Eqs. 20

and 24, it follows that

$$Y_1 = iY_0 \sum_{n=0}^{\infty} C_{0n} a_n / a_0 N_0. \quad (31)$$

In an isolated radiating system the only way in which a driving-point admittance may be defined is by the ratio of current to voltage at this point. Thus, in this system if we define an admittance Y_2 to be the ratio of the current at the foot of the antenna to the voltage applied between the ground and the foot, then it follows, from expressions which can easily be found for the current and the voltage, that

$$Y_2 = iY_0 b \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{nm} a_m \varphi_n(b) / a_0 N_n. \quad (32)$$

The first term of this series is the admittance Y_1 .

VI. The exact solution to this problem for general values of the quantities k , a , and b involves the solution of an infinite set of linear equations whose coefficients are infinite integrals. Even with the help of modern computing machinery the work would be exceedingly difficult. We shall first consider, however, the solution for small values of the parameter $\epsilon = k(a - b)$. If we put $\sigma = ka$, and $\rho = b/a$, then it is clear that ϵ may be small if either $\sigma \rightarrow 0$, $\rho \neq 1$ or if $\sigma \neq 0$, $\rho \rightarrow 1$: the first case corresponds to the physical problem of the antenna which is thin in comparison with the excitation wavelength, and the second to the problem of the thick antenna with a low impedance line feed. We will exclude, however, the case for which $\rho \rightarrow 1$ and $\sigma \rightarrow 0$, this being the case of the thin antenna with a low impedance feed.

For the limiting cases $\sigma \rightarrow 0$, $\rho \neq 1$, $\sigma \neq 0$, $\rho \rightarrow 1$ the magnitudes of the functions which appear in the infinite set of equations are derived in the Appendix. To solve the problem when $\epsilon \rightarrow 0$, we first assume that in the equations 24, $a_n = 0$ for $n \geq 1$. Then from the first equation of the set, it follows that

$$a_0 = -2ik, \quad \text{as } \rho \rightarrow 1, \quad \sigma \neq 0, \\ \text{or } \sigma \rightarrow 0, \quad \rho \neq 1.$$

If we take this first approximation, we find from the $(n + 1)$ th equation that

$$a_r = \frac{ip_r C_{r0}}{N} a_0, \\ = 0(1) \quad \text{as } \rho \rightarrow 1, \quad \sigma \neq 0, \quad \text{or } \sigma \rightarrow 0, \quad \rho \neq 1.$$

Using these results for the orders of magnitude of the a_r in the complete set of equations, we find that as $\epsilon \rightarrow 0$

$$a_0 = -2ik[1 + 0(\epsilon)], \\ a_r = 0(1), \quad r \geq 1.$$

We now find in the expressions for the physically interesting quantities that $a_0 = -2ik$ is the only coefficient to make a significant contribution. This value for a_0 is that which would arise in analysis of an open-circuited line ignoring end effects.

From Eq. 19, the unknown function may now be given in the form

$$L(r) = -2ik/r.$$

$$\text{ADMITTANCE } Y = G + iB \left[\epsilon = k(a-b), \rho = b/a, Y_0 = 2\pi\omega\epsilon_0/k \ln(a/b) \right].$$

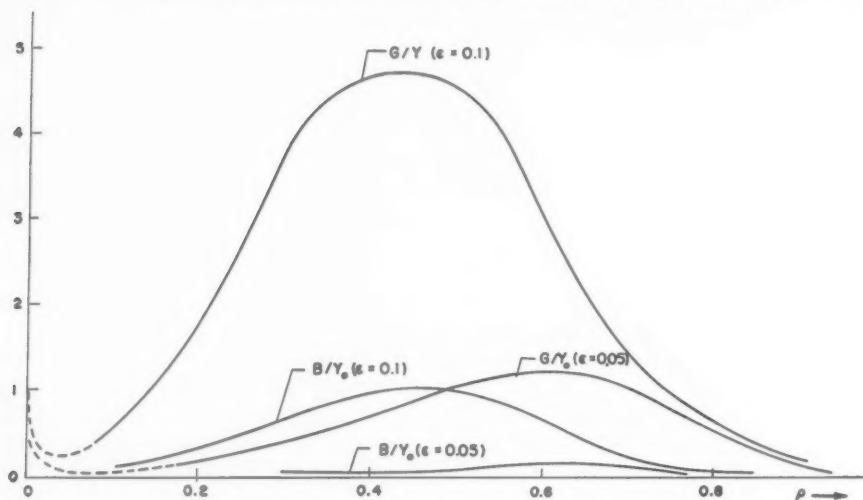


FIG. 2

The corresponding admittance of the radiating system, from either definition in Eq. 30 or 31 takes the form

$$Y = ikY_0C_{00}[1 + O(\epsilon)]/N_0.$$

The value of Y in the limit as $\sigma \rightarrow 0$, $\rho \neq 1$ is $O(\epsilon)$. The limit as $\rho \rightarrow 1$, $\sigma \neq 0$ does not exist because the second derivative of C_{00} with respect to ρ does not exist. In this case the strongest statement to be made is that the normalized admittance Y/Y_0 is zero.

Numerical values of the admittance for small values of ϵ are given in the accompanying graph (Fig. 2). The normalized admittance approaches unity as $\rho \rightarrow 0$ for fixed small values of σ . The open-circuit approximation used to calculate results is not therefore applicable for small values of ρ . From Equation 29, when $\epsilon \rightarrow 0$ we find that

$$P = \frac{2\omega}{\mu k^2 r^2} \frac{[J_0(\sigma \sin \varphi)Y_0(\sigma \rho \sin \varphi) - Y_0(\sigma \sin \varphi)J_0(\sigma \rho \sin \varphi)]^2}{[J_0(\sigma \rho \sin \varphi)]^2 + [Y_0(\sigma \rho \sin \varphi)]^2} [1 + O(\epsilon)].$$

For $r > a$ at large distances from the end of the line, where $\tan \varphi = r/z$ this expression may be simplified in certain cases.

In the case when $\sigma \rightarrow 0$, $\rho \neq 1$

$$P \approx 2\omega[\ln \rho]^2/\mu k^2 r^2 \{ [\ln(\sigma \rho \sin \varphi)/2]^2 + \pi^2/4 \}.$$

In the case when $\rho \rightarrow 1$, $\sigma \neq 0$ is more interesting. Thus

$$P \approx 8\omega(1 - \rho)^2/\mu k^2 r^2 \pi^2 \{ [J_0(\sigma \rho \sin \varphi)]^2 + [Y_0(\sigma \rho \sin \varphi)]^2 \},$$

and when $\sin \varphi \approx r/z$ for small enough values, $r > a$,

$$P \approx 8\omega(1 - \rho)^2/\mu k^2 r^2 \pi^2 \{ 4[\ln(\sigma r/2z)/\pi]^2 + 1 \}.$$

This expression shows that we may expect a behavior for P like $[1/r \ln r]^2$ for short wave-

lengths when $\sigma = ka = 0(z)$ for large values of z . More generally, if, when $\rho \rightarrow 1$ and $\sin \varphi$ is not small, we take values of σ which are large, but not so large that the relation $\sigma(1 - \rho) = \epsilon \rightarrow 0$ is not satisfied, we find that

$$P \approx 4\omega\epsilon(1 - \rho) \sin \varphi / \mu\pi k^2 r^2.$$

VII. In the general case, for $\epsilon > 0$, we shall prove two results. The first is

Theorem 1. The infinite set of equations given in Eq. 24 has for a solution at most one set of values a_n if $\epsilon = ka(ab) > 0$.

It will be recalled that a_n is a coefficient in the orthogonal expansion in $a > r > b$ of the function $r^{1/2} L(r)$.

Suppose that we have fixed $r\pi < \epsilon \leq \pi(r + 1)$ so that p_m ($0 \leq m \leq r$) is real and $ip_m = q_m$ ($m \geq r + 1$) is also real. The homogeneous set of equations corresponding to Eq. 24 is

$$kN_m a_m / ip_m = \sum_{n=0}^{\infty} C_{nm} a_n, \quad \text{for } m \geq 0. \quad (33)$$

Putting $\alpha_m + i\beta_m = a_m$, $C_{mn} = K_{mn} + iL_{mn}$ we find that

$$\begin{aligned} \sum_{n=0}^m (K_{mn}\alpha_n - L_{mn}\beta_n) &= kN_m \beta_m / p_m, \quad \text{for } m \leq r, \\ &= kN_m \alpha_m / q_m, \quad \text{for } m \geq r + 1. \\ \sum_{n=0}^{\infty} (K_{mn}\beta_n + L_{mn}\alpha_n) &= -kN_m \alpha_m / p_m, \quad \text{for } m \leq r, \\ &= kN_m \beta_m / q_m, \quad \text{for } m \geq r + 1. \end{aligned}$$

It follows that

$$\begin{aligned} -\beta_m \sum_{n=0}^m L_{mn}\beta_n - \alpha_m \sum_{n=0}^{\infty} L_{mn}\alpha_n &= 0, \quad \text{for } m \geq r + 1, \\ &= kN_m (\alpha_m^2 + \beta_m^2) / p_m, \quad \text{for } m \leq r, \end{aligned} \quad (34)$$

so that

$$k \sum_0^r N_m (\alpha_m^2 + \beta_m^2) / p_m = - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\beta_m L_{mn}\beta_n + \alpha_m L_{mn}\alpha_n).$$

Now

$$L_{mn} = 2k \int_0^k \frac{\varphi_m(a)\varphi_n(a)K^2 a^2 [J_0(Ka)Y_0(Kb) - Y_0(Ka)J_0(Kb)]^2}{(K - K_m)(K - K_n)\{[J_0(Kb)]^2 + [Y_0(Kb)]^2\}} dp,$$

so that the result of changing the orders of integration and summation in Eq. 34 is

$$\begin{aligned} \sum_{m=0}^r N_m (\alpha_m^2 + \beta_m^2) / p_m &= -2 \int_0^k \frac{dp K^2 a^2}{\{[J_0(Ka)]^2 + [Y_0(Ka)]^2\}} \\ &\quad \cdot \sum_0^{\infty} \left\{ \left(\frac{\beta_m \varphi_m(a)}{K^2 - K_m^2} \right)^2 + \sum_0^{\infty} \left(\frac{\alpha_m \varphi_m(a)}{K^2 - K_m^2} \right)^2 \right\}. \end{aligned}$$

The right-hand side of the equation is ≤ 0 . The left-hand side of this equation is, however, a non-negative quantity so that each side of the equation must be zero. It follows that

$$\alpha_m, \beta_m = 0 \quad \text{for all } m \geq 0.$$

Given the infinite set of equations, it is necessary to know whether an approximate solution can be found by solving only the first $N + 1$ equations for the unknowns ($n \leq N$), taking all the remaining unknowns to be zero. We shall therefore prove the following

Theorem 2. (i) The determinant of the finite set of equations

$$kN_m(2\delta_{0m} + a_m/ip_m) = \sum_{n=0}^N C_{nm}a_n, \quad 0 \leq m \leq N, \quad (35)$$

does not vanish for any positive value of ϵ

(ii) The unique solution of the finite number of equations is the set $\{a_n^N\}$, $0 \leq n \leq N$, and $\lim (a_n - a_n^N) = 0$ as $N \rightarrow \infty$.

The first part of the theorem is proved in the same way as in Theorem 1. For the second part we find from Eqs. 24 and 35 that

$$kN_m(a_m - a_m^N)/p_m = \sum_{n=0}^N C_{nm}(a_n - a_n^N) + \sum_{n=N+1}^{\infty} C_{nm}a_n.$$

Now from 19, using Parseval' Theorem, since $r^{1/2}L(r)$ is a function of integrable square in $a > r > b$, we see that

$$\int_b^a r[L(r)]^2 dr = \sum_0^{\infty} |a_n|^2 N_n = O(1), \quad \text{for } \epsilon > 0.$$

Since $N_n = O(r^{-2})$ as $n \rightarrow \infty$, a necessary condition for the convergence of this series is that as $n \rightarrow \infty$, for $\delta > 0$,

$$|a_n| = O(n^{1-\delta}).$$

It is shown in the Appendix that the infinite integral C_{mn} is absolutely convergent for all $\epsilon > 0$. As $n \rightarrow \infty$ $|C_{mn}| = O(n^{-3})$ so that

$$\sigma_m^N = \left| \sum_{n=N+1}^{\infty} C_{mn}a_n \right| = O(N^{-1-\delta})$$

if N is large enough, for all $m > 0$.

Thus

$$\left| \frac{kN_m(a_m - a_m^N)}{ip_m} - \sum_{n=0}^N C_{mn}(a_n - a_n^N) \right| = \sigma_m^N, \quad (36)$$

and since the determinant of the finite set of equations is non-zero, the solution of 36 must be of the form

$$a_m - a_m^N = O(\sigma_m^N).$$

Therefore, we may approach as close as we like to a solution of the infinite set of equations if we take N large enough, and from Theorem 1 this solution is a unique one.

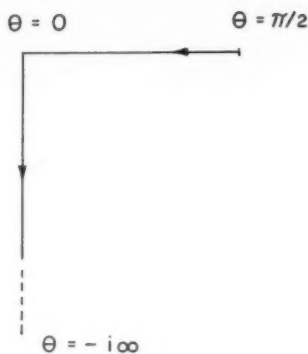


FIG. 3

Theorem 2 shows that we may obtain a useful approximation to the solution by solving for the first $N + 1$ unknowns, and we are able to use a variational method to hasten the convergence to a correct result. This approach may be made in a more general problem in which there may be set up any number of propagating modes within the coaxial line.

APPENDIX

We transform the integrand in Eq. 23 by putting $p = k \cos \theta$. To correspond to the branch of K , we must take the path of integration in the complex θ -plane as shown in the diagram (Fig. 3).

We are principally interested in the behavior of C_{mn} for small values of the quantity $\epsilon = k(a - b)$. We have to consider separately the possibility of the ratio $\rho = b/a$ approaching unity with $\sigma = ka$ not small, and also the case in which σ is small and ρ is not close to unity.

From the definition of the functions $\varphi_n(a)$ in Sec. III, it follows that as $\epsilon \rightarrow 0$

$$\begin{aligned} a\varphi_n(a) &= -2\epsilon/nk\pi^2, \quad \text{for } n \geq 1 \\ &= 1, \quad \text{for } n = 0. \end{aligned}$$

The functions N_n defined in the same section behave in the following manner for small values of ϵ . Thus, for any value of σ

$$N_0 = \ln \rho = \begin{cases} 0(1) & \text{if } \rho \rightarrow 1, \\ 0(\epsilon) & \text{if } \rho \rightarrow 1, \end{cases}$$

and

$$\begin{aligned} N_n &= 2\{1 - [J_0(K_n a)/J_0(K_n b)]^2\}/(\pi K_n)^2, \\ N_n &= \frac{2\epsilon^2}{\pi^2 k^2 n^2} \{1 - [J_0(n\pi/1 - \rho)/J_0(n\pi\rho/1 - \rho)]^2\}, \\ &= 0(\epsilon^2) \quad \text{if } \rho \neq 1 \quad \sigma \rightarrow 0, \\ &= 0(\epsilon^3) \quad \text{if } \rho \rightarrow 1 \quad \sigma \neq 0. \end{aligned}$$

We shall first consider the convergence of the integral C_{mn} . On the path C we have already established the bounded nature of the integrand near the origin, since we have examined its behavior in the p -plane near the branch point $p = +k$. This property is independent of the values of the parameters σ and ρ . On the negative imaginary axis the integrand I_{mn} is a real function involving modified Bessel functions of real argument.

We must examine the uniformity of convergence of the integral $\int_0^\infty I_{mn} d\varphi$ with respect to the parameters σ and ρ . Let us take a small positive number σ_0 as close as we like to zero. Then, for $0 < \rho \leq 1$, $\sigma \geq \sigma_0 > 0$, we may choose $\varphi_0 \gg \sinh^{-1}(1/\sigma_0)$, and for $\varphi_1 > \varphi_0$ we may use the asymptotic expansion of the modified Bessel functions. Since for all $n \geq 0$, $K_n \geq 0$, it follows that

$$\left| \int_{\varphi_1}^{\infty} I_{mn} d\varphi \right| \leq \frac{2}{\pi} \int_{\varphi_1}^{\infty} \frac{d\varphi}{\sinh \varphi} \left[N_n \delta_{mn} + \frac{a\varphi_n(a)a\varphi_m(a)}{2\sigma \sinh \varphi} (1 + O(\sigma \sinh \varphi)^{-1}) \right].$$

Since for $\theta > \varphi_1$, $\sigma_{0m} \theta > 1$, it follows from the convergence of the integral $\int_{\varphi_1}^{\infty} \text{cosech } \theta d\theta$ that C_{mn} is uniformly convergent with respect to φ and σ in the given range. This is true for all $m, n \geq 0$.

Now consider C_{00} . The integrand vanishes for $\sigma = 0$, $\rho \neq 1$ or $\sigma \neq 0$, $\rho = 1$. It is clear that for $\rho \neq 1$, σ is a factor of C_{00} , and for $\sigma = 0$, $(1 - \rho)$ is a factor. The derivative of the integrand with respect to ρ is uniformly integrable in the same range of parameters, and this vanishes also for $\rho = 1$; it follows that for $\sigma \neq 0$, $\rho \neq 1$, $C_{00} = O(\epsilon^2)$ while for $\rho \neq 1$, $\sigma \rightarrow 0$ $C_{00} = O(\epsilon)$. The second derivative with respect to ρ does not exist if $\rho = 1$, $\sigma \neq 0$, nor does the first derivative with respect to σ when $\sigma = 0$, $\rho \neq 1$.

The magnitude of C_{mn} for at least one of m or n not zero is determined by the magnitude of the integrand on the path C at a finite distance from the origin. Since $K_n \sim n\pi k/\epsilon$, we find that

$$\begin{aligned} C_{0n} = C_{n0} &= O(\epsilon^4) \quad \text{for } \rho \rightarrow 1, \quad \sigma \neq 0, \\ &= O(\epsilon^3) \quad \text{for } \rho \neq 1, \quad \sigma \rightarrow 0, \end{aligned}$$

and that for both m and n not zero,

$$\begin{aligned} C_{mn} &= O(\epsilon^5) \quad \text{for } \rho \rightarrow 1, \quad \sigma \neq 0, \\ &= O(\epsilon^4) \quad \text{for } \sigma \rightarrow 0, \quad \rho \neq 1. \end{aligned}$$

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—NOTES—

ON THE WORK OF A FORCE CROSSING A BEAM *

By L. MAUNDER (*University of Edinburgh*)

1. **Introduction.** Timoshenko [1] has discussed the paradox that during its transit across an elastic beam whose ends are supported at the same horizontal level, as shown in Fig. 1, a vertical force apparently performs no net work, yet leaves the beam in a state of oscillation. His explanation, however, refers to a different but closely related problem, in which the force is always directed at right-angles to the deflected beam. A horizontal force component is thereby introduced which evidently is capable of doing work.

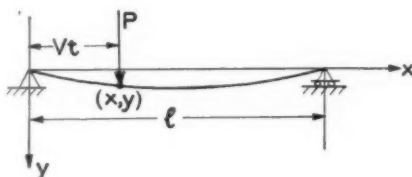


FIG. 1

Lee [2] shows that this work equals the sum of the strain and kinetic energies possessed by the beam on departure of the force. But, pointing out that this result does not explain the original difficulty, he returns to the case of the purely vertical force and shows that, contrary to appearances, the work which it does on the beam is not zero, but equals the energy retained by the beam. This result then resolves, in principle, the original paradox.

In view of the usual interpretation of the work of a force, however, it remains curious that the purely vertical force does positive work during transit between points on the same horizontal level. It therefore seemed worthwhile to consider the problem further, not only because of the physical mechanisms involved, but also because of its bearing on the definition of work.

2. **Work relations.** As in [2], we define the rate at which a force does work on a body as the scalar product of the force vector and the velocity vector of the particle of the body at the point of application of the force. This definition permits association of the work done on the body with the change in its kinetic, strain and potential energy. But it does not always describe fully the transmission of work from one physical system to another, as may be illustrated by the system of Fig. 2.

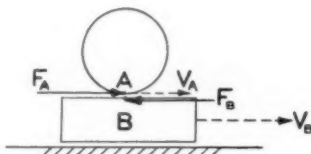


FIG. 2

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Block B slides on a frictionless horizontal plane with velocity V_B : the center of the circular disc moves horizontally only and the disc rotates such that the velocity of particle A in contact with the block is V_A . As the normal components of the contact forces between disc and block do no work, only the frictional components F_A and F_B are shown. From the preceding definition we find that F_B does work on the block at the rate $(-F_B V_B)$ and that F_A does work on the disc at the rate $(F_A V_A)$. This description can usefully be expanded, however, by stating that F_B extracts work from the block at the rate $(F_B V_B)$, of which part is transmitted to the disc at the rate $(F_A V_A)$ and part is transformed into heat by the force pair $F(F_A = F_B = F)$ at the rate $F(V_B - V_A)$.

If V_A is zero, when the figure might represent metal-cutting with a fixed tool at A , work is extracted by the contact force from the traveling work-piece B at the rate $(F V_B)$. All of the extracted work is transformed by the contact force pair and no work is done on the fixed tool. If $V_A = V_B = V$, so that the disc rolls on the block, work is extracted by the contact force from one body at the rate $(F V)$ and is transferred entirely to the other. In this case, no work is transformed into heat.

In general the flow of work from one system to another should be studied through the action of the force pair or pairs exerted between the systems. When tangential forces occur at moving surfaces, the foregoing definition of work can lead to the result that the work extracted from one system differs from that transmitted to the other. In certain cases, the difference is transformed into energy associated with microscopic phenomena, such as thermal energy.

3. The physical system. The application of a purely vertical force to the deflected beam requires that the surface of the beam be capable of sustaining tangential forces. They may be associated with sliding or rolling. We consider both possibilities.

A particle of negligible mass can slide across the beam in the prescribed way under the action of an applied constant vertical force and the equal and opposite reaction of the beam, assuming that the frictional properties of the surfaces allow the latter to be developed. The total work done on the particle by the applied force is expressed by the integral

$$P \int_0^{t/v} \left(\frac{\partial y}{\partial t} + V \frac{\partial y}{\partial x} \right)_{x=vt} dt \quad (1)$$

in which the integrand is the vertical component of the velocity of the particle. As the net vertical displacement of the particle is zero, the work (1) is zero. Since the particle stores no work, the total work extracted from it by the contact force applied by the beam must also be zero. Now the (zero) extracted work is partly transferred to the beam and partly transformed into heat by the frictional components of the contact force pair. The net work done on the beam is expressed [2] by the integral

$$P \int_0^{t/v} \left(\frac{\partial y}{\partial t} \right)_{x=vt} dt \quad (2)$$

which, being equal to the energy retained by the beam, is necessarily positive. Hence the work transformed into heat, namely,

$$P V \int_0^{t/v} \left(\frac{\partial y}{\partial x} \right)_{x=vt} dt \quad (3)$$

must be negative, from which we conclude that the assumed frictional characteristics

cannot exist even in principle. The detailed explanation is that the assumptions entail negative friction, i.e. friction tending to increase the relative velocity between particle and beam, when $(\partial y/\partial x)_{x=v_t}$ is negative. Sliding friction must therefore be discarded as a possibility.

The alternative is rolling contact, of which the simplest case is the rolling of a circular disc of negligible mass and radius r . Its point of contact with the beam will have the prescribed horizontal velocity V if the velocity of its center has horizontal and vertical components V and $[(\partial y/\partial t) + V(\partial y/\partial x)]_{x=v_t}$ respectively, and if the disc has a clockwise angular velocity

$$(V/r)[1 + (\partial y/\partial x)^2]_{x=v_t}^{1/2}.$$

This motion requires that certain forces be applied to the disc in addition to the prescribed vertical reaction P of the beam. They are equivalent to a vertical force P applied at the center of the disc together with a counterclockwise couple of magnitude $Pr\{(\partial y/\partial x)/[1 + (\partial y/\partial x)^2]^{1/2}\}_{x=v_t}$.

As the disc stores no work, the work done on it by the applied central force and the applied couple must be extracted fully by the contact force of the beam. Moreover, the work so extracted is transferred entirely to the beam, since no work is transformed at a rolling contact, as noted previously. Now the applied central force and the applied couple do work on the disc at the rates

$$P[(\partial y/\partial t) + V(\partial y/\partial x)]_{x=v_t} \quad \text{and} \quad -PV(\partial y/\partial x)_{x=v_t} \quad (4)$$

respectively, as follows from definition and the prescribed motion of the disc. Hence the rate at which work is transferred by the contact force to the beam is the sum of the rates (4), namely,

$$P(\partial y/\partial t)_{x=v_t}. \quad (5)$$

If we regard the central vertical force and the couple as external forces which are applied to the system of disc and beam, it is noteworthy that the total work done on the system by the central force is zero. The work done on the system by the couple equals the work done on the beam by the contact force, i.e.

$$-PV \int_0^{1/V} \left(\frac{\partial y}{\partial x} \right)_{x=v_t} dt = P \int_0^{1/V} \left(\frac{\partial y}{\partial t} \right)_{x=v_t} dt. \quad (6)$$

4. Conclusion. A full description of the work of a force requires specification of the physical nature of the systems concerned. A physical inconsistency appears if the vertical force is considered to be applied to the beam through a sliding body, but it can be applied through a rolling body. In the simplest case of a rolling circular disc of negligible mass, the external forces which must be applied to the disc are equivalent to a constant vertical force applied at the center of the disc and a couple. The vertical force does no net work on the disc during transit of the beam, and the couple does an amount of work on the disc equal to that retained by the beam.

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AN IMPLICIT, NUMERICAL METHOD FOR SOLVING THE n-DIMENSIONAL HEAT EQUATION*

By GEORGE A. BAKER, JR. (*Los Alamos Scientific Laboratory, Los Alamos, New Mexico*)

1. Introduction. The work of this paper represents an extension of the work of Baker and Oliphant [1] to the n -dimensional, linear, heat-flow problem with arbitrary spacing between the mesh points. We give an implicit scheme which we are able to solve exactly, and we prove that it is unconditionally stable. We treat only the linear case because the generalization to the non-linear case may be treated in precisely the same way as was done by Baker and Oliphant and the same conclusions hold in the n -dimensional case concerning this generalization as in the two-dimensional case.

2. The difference approximation. The basic partial differential equation we wish to consider is

$$\nabla^2 \theta = S(\mathbf{x}, t) + A^2 \theta + B^2 \frac{\partial \theta}{\partial t}, \quad (2.1)$$

where ∇^2 is the n -dimensional Laplacian with respect to \mathbf{x} , \mathbf{x} is an n -dimensional position vector, t is time, and S is a source function. Not both A and B may be zero. By setting $B = 0$ ($A \neq 0$) we may solve at one step for the asymptotic solution of (2.1). We wish to approximate this differential equation by an implicit, difference scheme to allow approximate numerical calculation of the function θ . We shall define our mesh by n sequences

$$0 < x_{1k} < \cdots < x_{I_k} < x_{I_k+1} < \cdots < x_{M_k} < L_k, \quad k = 1, \dots, n \quad (2.2)$$

which are the values of the coordinates of the mesh points in an n -dimensional vector space.

Let us define the matrices

$$D_{J_k}^{I_k} = \frac{2 \delta_{J_k}^{I_k-1}}{(x_{J_{k+1}} - x_{J_k})(x_{J_{k+1}} - x_{J_{k-1}})} - \frac{2 \delta_{J_k}^{I_k}}{(x_{J_{k+1}} - x_{J_k})(x_{J_k} - x_{J_{k-1}})} + \frac{2 \delta_{J_k}^{I_k+1}}{(x_{J_k} - x_{J_{k-1}})(x_{J_{k+1}} - x_{J_{k-1}})}, \quad k = 1, n, \quad (2.3)$$

where δ_i^j is the Kronecker delta. Let us approximate

$$\frac{\partial \theta}{\partial t} = [3\theta(t) - 4\theta(t - \Delta t) + \theta(t - 2\Delta t)]/(2\Delta t) \quad (2.4)$$

and define

$$\beta = -[A^2 + 3B^2/(2\Delta t)]. \quad (2.5)$$

We may now consider the n th order tensor (suppressing the various Kronecker δ 's)

$$\begin{aligned} \sum_{I_k} \beta^{1-n} \prod_{k=1}^n (\beta + D_{J_k}^{I_k}) \theta_{I_1 \dots I_k \dots I_n} \\ = \sum_{I_k} \left[\beta + \sum_{k=1}^n D_{J_k}^{I_k} + \beta^{-1} \sum_{i < k}^n D_{J_k}^{I_k} D_{J_i}^{I_i} + \cdots \right] \theta_{I_1 \dots I_k \dots I_n}. \end{aligned} \quad (2.6)$$

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It is evident that

$$\sum_{j_k} D_{j_k}^{I_k} \theta_{I_k} \text{ represents } \frac{\partial^2 \theta}{\partial x_k^2} \quad (2.7)$$

to within terms of the second order in Δx_k . Hence (2.6), neglecting 4th and higher order derivatives, represents to within second order in Δx_k the left hand side of

$$\beta \theta + \nabla^2 \theta = S(x, t) + A^2 \theta + B^2 \frac{\partial \theta}{\partial t} + \beta \theta. \quad (2.8)$$

By definition β has been chosen so that the right-hand side of (2.8) is independent of $\theta(t)$ and depends only on $\theta(t - \Delta t)$, $\theta(t - 2\Delta t)$, and a known function, S . We can solve the implicit equations (2.6) for $\theta_{I_1, \dots, I_k, \dots, I_n}$, as $\beta \delta_{j_k}^{I_k} + D_{j_k}^{I_k}$ is a triple diagonal matrix. We can factor it as

$$\beta \delta_{j_k}^{I_k} + D_{j_k}^{I_k} = \sum_{L_k} w_{j_k}^{L_k} b_{L_k}^{I_k}, \quad (2.9)$$

where w is a lower triangular matrix. The matrix elements are given by the recursion relations

$$\begin{aligned} w_{j_k}^{L_k} &= 0 \quad \text{if } j_k - L_k \neq 0, 1 \\ b_{L_k}^{I_k} &= 0 \quad \text{if } I_k - L_k \neq 0, 1 \\ b_{I_k}^{I_k} &= 1 \\ w_{j_k}^{j_k-1} &= D_{j_k}^{j_k-1} \\ w_{j_k}^{j_k} &= \beta + D_{j_k}^{j_k} - w_{j_k}^{j_k-1} b_{j_k-1}^{j_k} \quad (b_0^1 = 0) \\ b_{j_k}^{j_k+1} &= D_{j_k}^{j_k+1} / w_{j_k}^{j_k}. \end{aligned} \quad (2.10)$$

Let us now define

$$g_{L_1, \dots, L_k, \dots, L_n} = \sum_{I_k} \left(\prod_{k=1}^n b_{L_k}^{I_k} \right) \theta_{I_1, \dots, I_k, \dots, I_n}. \quad (2.11)$$

Then, using the difference approximations (2.4) and (2.6), we may represent (2.1) by

$$\sum_{L_k} \beta^{1-n} \left(\prod_{k=1}^n w_{j_k}^{L_k} \right) g_{L_1, \dots, L_k, \dots, L_n} = a_{j_1, \dots, j_k, \dots, j_n}, \quad (2.12)$$

where the a 's are known quantities. Due to the triangular nature of the w 's we may proceed from low index numbers to high ones and solve (2.12) for the g 's by straightforward elimination. Then due to the triangular nature of the b 's we may proceed from high index number to low ones, and solve (2.25) for the θ 's by straightforward elimination.

3. Stability. We shall proceed with a proof of the unconditional stability of the method described in Sec. 2 by a method closely related to those employed by Baker and Oliphant [1]. We consider only the problem with S independent of t . As Eq. (2.1) is linear it will be sufficient to consider the solutions of the difference equation analogue of

$$\nabla^2 \theta = A^2 \theta + B^2 \frac{\partial \theta}{\partial t} \quad (3.1)$$

with homogeneous boundary conditions. This equation results from the subtraction of

the asymptotic solution from the difference equation analogue of (2.1). Let us seek a solution of the difference equation analogue of (3.1), such that, for some Z_v ,

$$\theta(\nu, \mathbf{x}, t) = \psi(\nu, \mathbf{x}) Z_v^{t/\Delta t}. \quad (3.2)$$

Equation (3.1) as expressed by (2.4), (2.5), and (2.6) is for a component (3.2)

$$\begin{aligned} \sum_{I_k} \left[\beta^{1-n} \prod_{i=1}^n (\beta \delta_{J_k}^{I_k} + D_{J_k}^{I_k}) - \beta \prod_{i=1}^n \delta_{J_k}^{I_k} \right] \theta_{I_1, \dots, I_k, \dots, I_n}(\nu) \\ = \{A^2 + B^2[3 - 4Z_v^{-1} + Z_v^{-2}]/(2\Delta t)\} \theta_{J_1, \dots, J_k, \dots, J_n}(\nu). \end{aligned} \quad (3.3)$$

Let us define

$$A^2 + \frac{B^2}{\Delta t} (3/2 - 2Z_v^{-1} + 1/2Z_v^{-2}) = \lambda \quad (3.4)$$

and an operator Δ_k , such that

$$\Delta_k \omega_{I_1, \dots, I_k, \dots, I_n} \equiv \frac{\omega_{I_1, \dots, I_k, \dots, I_n} - \omega_{I_1, \dots, I_{k-1}, \dots, I_n}}{x_{I_k} - x_{I_{k-1}}}. \quad (3.5)$$

Further let

$$\begin{aligned} F \equiv \sum_{I_k} \left\{ \sum_{k=1}^n (\Delta_k \theta_{I_1, \dots, I_k, \dots, I_n})^2 - \beta^{-1} \sum_{i < k} (\Delta_i \Delta_k \theta_{I_1, \dots, I_k, \dots, I_n})^2 \right. \\ \left. + \beta^{-2} \sum_{i < j < k} (\Delta_i \Delta_j \Delta_k \theta_{I_1, \dots, I_k, \dots, I_n})^2 - \dots \right\} \end{aligned} \quad (3.6)$$

and

$$G \equiv \sum_{I_k} (\theta_{I_1, \dots, I_k, \dots, I_n})^2 = 1. \quad (3.7)$$

One can now show that the solution of (3.3) is the same problem as finding the extrema of $F + \lambda G$ subject to (3.7), and homogeneous boundary conditions. As we have

$$\prod_{k=1}^n M_k = M \quad (3.8)$$

mesh-points, the theory of quadratic forms [2] assures us that since β is negative, there exists a complete set of m orthogonal vectors $(\theta_{I_1, \dots, I_k, \dots, I_n}(\nu))$ which satisfy (3.6) and (3.7) and therefore (3.3). As λ can be shown to be the negative of F , we must have

$$(3 - 4Z_v^{-1} + Z_v^{-2})/3 = \Delta t(\lambda - 2A^2/3)/B^2 < 0. \quad (3.9)$$

It easily follows [1] from (3.9) that for both roots

$$|Z_v| < 1. \quad (3.10)$$

Hence, the amplitude of all the $\theta(\nu)$ solutions decays to zero as time goes to infinity. Since the $\theta(\nu)$ form a complete orthonormal set, any initial value problem can be expanded in terms of them, and hence we may conclude on the basis of (3.10) that the method is unconditionally stable.

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HELICAL FLUID FLOWS*

By ROBERT H. WASSERMAN (*Michigan State University*)

Introduction. Potential helical flows have been completely described by G. Hamel [1]. Nemenyi and Prim, and N. Coburn have obtained some Beltrami helical flows [2, 3]. A simple description will be given here of all steady incompressible helical flows and all steady compressible helical flows with entropy constant along stream lines.

The equations of helical flow. The class of compressible flows to be considered here are those governed by the following differential equations in which v^* is the velocity, p is the pressure, ρ is the density, S is the entropy, and t^* is the unit tangent vector along the stream lines;

$$\nabla \cdot \rho v^* = 0 \text{ (continuity equation)} \quad (1)$$

$$(\rho v^* \cdot \nabla) v^* = -\nabla p \text{ (equation of motion)} \quad (2)$$

$$\rho = \rho(p, S) \text{ (equation of state)} \quad (3)$$

$$t^* \cdot \nabla S = 0 \text{ (entropy is constant along stream lines)} \quad (4)$$

For incompressible flows we must satisfy Eqs. (1) and (2) and the special case $\rho = \text{constant}$ of Eq. (3).

Now we assume that the flows are helical; i.e., the stream lines of the flow are parallel helices on coaxial circular cylinders. Such flows have the property $\nabla \cdot t^* = 0$. This may be seen immediately if we introduce cylindrical coordinates r, θ, z and decompose t^* according to

$$t^* = \sin \beta \theta^* + \cos \beta z^*.$$

Note that the angle β of the helices is in general a function of r .

With the condition $\nabla \cdot t^* = 0$ the continuity equation reduces to

$$t^* \cdot \nabla \ln \rho q = 0, \quad (5)$$

where q is the magnitude of the velocity. Also, from Eqs. (3) and (4)

$$t^* \cdot \nabla \rho - \frac{\partial \rho}{\partial p} t^* \cdot \nabla p = 0. \quad (6)$$

Finally, we write Eq. (2) in intrinsic form [4]:

$$t^* \cdot \nabla p = -\rho t^* \cdot \nabla \frac{q^2}{2}, \quad (7)$$

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$$n^* \cdot \nabla p = -\rho q^2 \kappa, \quad (8)$$

$$b^* \cdot \nabla p = 0, \quad (9)$$

where n^* is the unit principal normal vector of the stream lines, b^* is the unit binormal vector of the stream lines, and κ is the curvature of the stream lines. Then Eqs. (6), (7), and (8) give us the result that either

$$(a) \quad t^* \cdot \nabla q = t^* \cdot \nabla p = t^* \cdot \nabla \rho = 0$$

or

$$(b) \quad q^2 = \frac{\partial p}{\partial \rho} \quad \text{and} \quad p = B(S) - \frac{A(S)}{\rho},$$

where the functions $A(S)$ and $B(S)$ are restricted only by the condition $A(S) > 0$. Clearly, the alternative (b), in which the equation of state includes that used by Chaplygin, and Kármán and Tsien, only occurs in the compressible case.

The fact that the helices are geodesics on the cylinders $r = \text{const.}$ means that n^* is normal to these cylinders. With this, and using Euler's equation for the normal curvature of a curve to evaluate κ , Eq. (8) becomes

$$\frac{\partial p}{\partial r} = \rho q^2 \frac{\sin^2 \beta}{r}. \quad (10)$$

The two classes of helical flows. In case (a) Eq. (10) reduces to

$$\frac{dp}{dr} = \rho q^2 \frac{\sin^2 \beta}{r}.$$

If in this equation it is understood that ρ and q are constant along stream lines (i.e., condition (a) holds), then it embodies all the conditions for helical flows in case (a). That is, there is a helical flow corresponding to any set of functions p , ρ , q , β satisfying this equation.

In case (b) if we introduce the coordinates

$$\alpha = z + \theta r \tan \beta,$$

$$\psi = z - \theta r \cot \beta,$$

$$r = r,$$

then Eq. (10) may be written

$$r \frac{\partial p}{\partial r} + (\alpha - \psi) \sin \beta \cos \beta \frac{d}{dr} (r \tan \beta) \frac{\partial p}{\partial \alpha} = (B - p) \sin^2 \beta.$$

If in this equation it is understood that p is only a function of r and α , and B is only a function of r and ψ then any solution gives a helical flow. As an example of a family of solutions we have those in which

$$B = \text{const.}$$

$$r \tan \beta = \lambda = \text{const.}$$

$$p = B - \frac{(\lambda^2 + r^2)^{1/2}}{r} F(\alpha),$$

where F is an arbitrary positive function of α .

Some general properties of flows of case (a). In case (a) which applies to all fluids except those having the special form of the equation of state of case (b) our basic equation is very simple and we can readily obtain some general properties of helical flows from it.

Thus we see that

I. p is a non-decreasing function of r , and it is easy to construct examples for which it is either bounded or unbounded.

II. ρ , q , and β may, in general, either increase or decrease with r . Moreover, ρ and q may vary from stream line to stream line on a cylinder as long as ρq^2 is constant on cylinders.

III. In certain important special cases the variation of ρ , q , and β is restricted. Thus, for example, in the incompressible case if the Bernoulli function is constant then q must decrease with r . In the compressible case, if the entropy and the stagnation enthalpy are constant, then q must decrease with r .

IV. For a polytropic gas our basic equation can be written quite simply in terms of the Mach number. We find that the Mach number may in general either increase or decrease with r . However, if the entropy and the stagnation enthalpy are constant then the Mach number must decrease with r .

V. With respect to the vorticity we note in particular

- (i) in contrast to the other quantities of the flow [in case (a)] the vorticity *can* vary along the stream lines.
- (ii) when the vorticity vector is normal to the stream lines the only possible stream line pattern is given by $\cot \beta/r = \text{const.}$ which is that of the simplest helical flow; that obtained by normal superposition of a potential vortex and a uniform rectilinear flow.

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A NOTE ON THE PAPER OF MILLER, BERNSTEIN AND BLUMENSON*

By F. A. J. FORD (*Admiralty Research Laboratory, Teddington, England*)

In connection with the above authors' recent paper [1] on generalized Rayleigh distributions, it may be of interest to show how the cumulative distribution function of the first order distribution of such a process can be expressed in terms of tabulated functions. The cumulative distribution function in question is

$$\int_0^R \frac{A}{\psi_0} \left(\frac{R}{A} \right)^{N/2} \exp [-(R^2 + A^2)/2\psi_0] I_{(N-2)/2} \left(\frac{RA}{\psi_0} \right) dR,$$

which may be written as

$$\frac{(2\psi_0)^{N/2}}{A^n} p_n \left\{ \frac{R}{(2\psi_0)^{1/2}}, \frac{A}{(2\psi_0)^{1/2}} \right\},$$

where

$$p_n(x, y) = \int_0^x 2u^{n+1} \exp [-(u^2 + y^2)] I_n(2uy) du \quad n = (N - 2)/2.$$

When N is odd the modified Bessel function I_n can, as is pointed out in [1], be expressed in terms of Hyperbolic functions, so that $p_n(x, y)$ can be expressed in terms of Error functions. On the other hand, if N is even (so that n is integral), $p_n(x, y)$ can still be expressed in terms of tabulated functions.

For, on integration by parts, it follows that

$$p_n(x, y) = yp_{n-1}(x, y) - x^n \exp [-(x^2 + y^2)] I_n(2xy) \quad (n \geq 1).$$

Thus

$$\begin{aligned} \sum_{r=0}^{n-1} [y^r p_{n-r}(x, y) - y^{r+1} p_{n-1-r}(x, y)] &= p_n(x, y) - y^n p_0(x, y) \\ &= -\exp [-(x^2 + y^2)] \sum_{r=0}^{n-1} x^{n-r} y^r I_{n-r}(2xy). \end{aligned}$$

This result may be written

$$p_n(x, y) = y^n p_0(x, y) - \exp [-(x^2 + y^2)] \sum_{s=1}^n x^s y^{n-s} I_s(2xy). \quad (1)$$

Now $p_0(x, y)$ has already been tabulated (a list of tables is given in [2], so that by using Eq. (1) numerical values of $p_n(x, y)$ can be conveniently obtained, at least for small values of n , from the tables of $p_0(x, y)$ and of the modified Bessel functions.

The author is indebted to the reviewer for suggesting a simpler derivation of Eq. (1) than that originally presented.

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*Received November 20, 1958.

Correction to my paper

**THE RELATION BETWEEN THE FLOW OF NON-NEWTONIAN FLUIDS
AND TURBULENT NEWTONIAN FLUIDS**

Quarterly of Applied Mathematics, XV, 212-215 (1957)

By R. S. RIVLIN (*Brown University*)

The last sentence should read:

"Certainly it appears quite likely that a phenomenological theory of the type considered, in which the stress in an element of the turbulent fluid (large compared with the eddy dimensions) is supposed to depend only on the kinematic variables in that element or on the velocity-gradient history of that element, will *not* be entirely adequate as a complete description of the flow properties of the turbulent fluid, since eddies can diffuse from one point of the fluid to another."

BOOK REVIEWS

Analysis of industrial operations. Edited by Edward H. Bowman and Robert B. Fetter. Richard D. Irwin, Inc., Homewood, Illinois, 1959. xi + 485 pp. \$7.95.

A collection of 27 papers (published, in the main, within the last four years) describing actual applications of quantitative methods to the analysis of industrial operating problems. This material was collected for use in a graduate course in the MIT School of Industrial Management. There is a bias in the selection of papers towards discussions of the fitting of mathematical models to actual problems, rather than the development of new models or new methods.

The reader is presumed to have a working knowledge of differential calculus and analytical statistics, and some understanding of linear programming, waiting line theory and the techniques of Monte Carlo simulation.

The papers are divided into five classifications: Applications of linear programming; Other programming applications; Waiting line applications; Applications of incremental analysis; Total cost and value models.

BRUCE CHARTRES

Elementary decision theory. By Herman Chernoff and Lincoln E. Moses. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1959. xv + 364 pp. \$7.50.

The day of determinism is past, even in the field of engineering, and it is rather amusing to remark that the determinist has inevitably brought it on himself. By constructing more and more realistic models of the physical universe, of greater and greater complexity, he has slowly forced himself into a position where the only escape from stalemate lies in the introduction of new rules into the game. The result is that we must coat ignorance of true cause and effect and inability to handle the resultant equations, even if we knew them, by the soothing, albeit sophisticated, salve of probability theory.

The modern engineer who is interested in feedback control processes, in automation, in processes involving adaptive control, and in any number of significant uses of digital and analogue computers, must acquaint himself with the mathematical theory of decision processes. He must understand the difficulties and advantages of uncertainty, and be familiar with the various tricks that are used to construct precise mathematical models of imprecise situations.

Unfortunately, many of the standard works in the field of decision-making start from a high mathematical level and continue upward. The result is that they are not suited to the needs of the beginner.

The book by Chernoff and Moses is a book designed specifically for the scientist interested in decision processes, but possessing only a rudimentary mathematical background. The authors have succeeded admirably in what they have set out to do. Carefully and slowly, they illustrate how mathematical models of decision-making under uncertainty are formulated. They discuss the choice of a criterion function, the choice of possible policies, the description by means of state variables, and so on.

All of this is done in a very pleasant, readable style, with numerous examples, and much important discussion. Most important, they are honest with the reader in constantly pointing out pitfalls and limitations. Their presentation illustrates very clearly that fundamental ideas can be transmitted without the pseudo-abstraction that befogs the Bourbaki and their devotees.

In the second half of the book, they give an introduction to classical statistics. Throughout, their aim is to follow the guiding idea of Wald who conceived of statistics as decision-making under uncertainty. In their presentation, they have followed the teaching of the late M. A. Girshick.

The book is recommended to mathematicians, physicists, engineers, and so forth, who wish to obtain a relatively painless introduction to this modern field, either for their own studies, or merely to keep abreast of current intellectual activity.

RICHARD BELLMAN

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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Title: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Words: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The distance between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter i and the prime ('), between alpha and a, kappa and k, mu and u, nu and n, eta and e.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set above letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which follow the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\quad}$. Compound exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponents with lengthy or complicated exponents the symbol \exp should be used, particularly if such exponents appear in the body of the text. Thus,

$$\exp [(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2 + b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos (\pi r/2b)}{\cos (\pi a/2b)} \text{ is preferable to } \frac{\cos \frac{\pi r}{2b}}{\cos \frac{\pi a}{2b}}$$

DIFFERENTIAL EQUATIONS

By Edwin P. Davis, Cornell University, in paper 7449

With a form, the differential equation is solved by the method of successive approximations, that they will find in the literature. The method is described in detail, and a list of references is given. The method is applied to the solution of the differential equation $y'' + y = 0$, and the results are compared with those obtained by the method of successive approximations. The method is also applied to the solution of the differential equation $y'' + y = 1$, and the results are compared with those obtained by the method of successive approximations. The method is also applied to the solution of the differential equation $y'' + y = x$, and the results are compared with those obtained by the method of successive approximations.

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ADVANCED STUDIES IN MATHEMATICS

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